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The totally geodesic Radon transform on the Lorentz space of curvature −1

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Abstract. We present a rotational symmetric model for the Lorentzian of curvature -1 in which the geodesics are straightlines. Investigating the Radon transform in this model yields to explicit formulas, inversion formulas range descriptions and support theorems.

1. Introduction

The Radon transform is heavily studied in a number of different settings nowadays. The spectrum of the investigations spread from the discrete case to the higher rank symmetric spaces but somewhat surprisingly the pseudo Riemannian spaces have not yet recieved much attention by now. The only works I know about are [3], [5].

We consider the isotropic Lorentz space $\mathcal{L}^n$ of signature $(1, n-1)$ with constant curvature -1, where n is the dimension of $\mathcal{L}^n$. This work depends basically on the observation that this space has a rotational $O(n)$ symmetry around its ideal points. (There are two ideal points and every timelike geodesic reach these points at $+\infty$ and $-\infty$.) The other crucial fact, that we use, is the geodesic correspondence between $\mathbb{R}^n$ and $\mathcal{L}^n$. Here, one has to note that a Lorentzian with geodesic correspondence is isotropic, hence harmonic and therefore of constant curvature according to Lichnerowicz and Walker (See [5]).

In Section 2 we present our model for $\mathcal{L}^n$. We use the quadratic hypersurface model of Helgason [5] and project it onto a hyperplane orthogonal to the rotational axis. We determine explicitly the arclength on the geodesics, and the Haar measure of the isotropy group at all the points.

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Using the results of Section 2, the third section gives explicitly the Radon transform for spacelike and timelike totalgeodesics, respectively. The dual Radon transform is also calculated according to the 'spacelike' and the 'timelike' part of the Lorentz sphere at the given point. Finally the spherical harmonic expansions of these transforms are shown.

In Section 4 we exhibit kind of intertwining operators between the Euclidean and the Lorentzian Radon transform. Using this connection, we invert the Lorentzian Radon transform on a certain space of even functions and give support theorem and range description. Then we prove that the spacelike Radon transform, and the timelike Radon transform in even dimensions, is also invertible.

The last section contains the necessary modification of the results of Section 4 for dimension two and considers the role of the odd functions that, contrary to the higher dimensions, are not annihilated in this case by the 'timelike' Radon transform.

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2. The model

We start with Helgason's quadratic hypersurface model [4], [5] for the Lorentz space of signature $(1, n-1)$ and of curvature -1, adapting also some of his notations. This is defined in $\mathbb{R}^{n+1}$ by the bilinear form

$$B(x,y) = x_1 y_1 - \sum_{i=2}^{n+1} x_i y_i \tag{2.1}$$

on the hypersurface $\mathcal{Q}^n_{-1}$ whose points satisfies

$$B(x,x) = -1. \tag{2.2}$$

These two equations give immediately, that $\mathcal{Q}^n_{-1}$ with its Lorentz structure is axially symmetric around the x_1-axis. We can also observe that it is symmetric with respect to the origin, and Helgason proved in [4], [5] that the geodesics are the (nonempty) intersections of $\mathcal{Q}^n_{-1}$ with the two dimensional subspaces of $\mathbb{R}^{n+1}$.

Our model $\mathcal{L}^n$ of the Lorentzian is the projection of $\mathcal{Q}^n_{-1}$ with its structure into the hyperplane $x_1 = 1$ through the origin of $\mathbb{R}^{n+1}$. Let $\mu\colon \mathcal{Q}^n_{-1} \to \mathcal{L}^n$ be this projection. By the above observations, the model $\mathcal{L}^n$ is rotational symmetric around the origin and the geodesics are straightlines. Obviously $\mathcal{L}^n$ is a double-covering model.

Let us take a point P on $\mathcal{Q}^n_{-1}$, and take the two dimensional subspace π of $\mathbb{R}^{n+1}$ containing P and the x_1-axis. Clearly, the intersection of π with $\mathcal{Q}^n_{-1}$ is a hyperbola (two sheeted). Say, this hyperbola intersects the subspace $x_1 = 0$ of $\mathbb{R}^{n+1}$ in the point O. Let r denote the Lorentzian distance of P and O. Then r is the Lorentzian distance of P from the equator, the intersection of $\mathcal{Q}^n_{-1}$ and $x_1 = 0$. Let the coordinates of P be $(p_1, p_2, \dots, p_{n+1})$ in $\mathbb{R}^{n+1}$. Relative to π we may use the coordinates $\rho_1 = p_1$ and $\rho_2 = \sqrt{\sum_{i=2}^{n+1} p_i^2}$. These coordinates are functions of r. Since r is arclength parameter

$$\left(\frac{d\rho_1}{dr}\right)^2 - \left(\frac{d\rho_2}{dr}\right)^2 = +1 \quad \text{and} \quad \rho_1^2 - \rho_2^2 = -1 \tag{2.3}$$

by (2.1) and (2.2). The second part gives a function $\rho(r)$ so that $\rho_1 = \sinh(\rho)$ and $\rho_2 = \cosh(\rho)$. Then the first part gives $d\rho/dr \equiv 1$, that is

$$p_1 = \rho_1 = \sinh(r) \quad \text{and} \quad \rho_2 = \cosh(r). \tag{2.4}$$

Hence the projection $\mu(P)$ of P onto $x_1 = 1$ is a point in the corresponding direction, i.e. in π, and $|\mu(P)| = \coth r$. We parameterize $\mathcal{L}^n$ so that (ω, r) means the point $\mu(P) = \omega \coth r$ in $x_1 = 1$, where $\omega \in S^{n-1}$, $\langle .,. \rangle$ is the standard Euclidean inner product and $\langle P, \omega \rangle > 0$.

Let $\langle .,. \rangle_{(\omega,r)}$ be the Lorentzian inner product on $T_{(\omega,r)}\mathcal{L}^n$ so that $\mu(P) = \omega \coth r$.

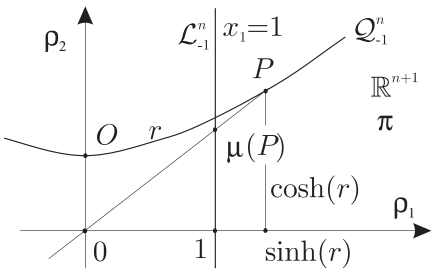

Figure 1.

To determine $\langle ., \rangle_{(\omega,r)}$ we observe Figure 1. It shows the situation what is seen on the plane π where $T_P\mathcal{Q}^n_{-1}$ is the tangent space of $\mathcal{Q}^n_{-1}$ at the point P. Obviously a vector v in $T_P\mathcal{Q}^n_{-1}$ orthogonal to π will be mapped by $\mu^*\colon T_P\mathcal{Q}^n_{-1} \to T_{(\omega,r)}\mathcal{L}^n$,

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the induced map of μ, into a vector $\mu^*(v)$ having the same Euclidean direction. Clearly $|\mu^*(v)|\sinh r = |v|$ in Euclidean meaning that, according to (2.1), coincides with the Lorentzian meaning in these directions. If v is in $\pi \cap T_P\mathcal{Q}^n_{-1}$ then $\mu^*(v)$ is in $\pi \cap T_{(\omega,r)}\mathcal{L}^n$ and it is orthogonal to the x_1-axis. It is easy to see that the Euclidean angle α of $T_P\mathcal{Q}^n_{-1}$ and the x_1-axis satisfies $\tan\alpha = \tanh r$. Therefore $|v| = |\mu^*(v)| \cdot \sinh r \cdot \sin\alpha/\cos 2\alpha$ giving

$$B(v,v) = |v|^2 \cdot \cos 2\alpha = |\mu^*(v)|^2 \cdot \sinh^4 r.$$

(A more formal way to prove this is to differentiate μ.) We can now conclude that

$$\langle (dr, d\omega), (dr', d\omega')\rangle_{(\omega,r)} = \sinh^2 r (dr dr' \sinh^2 r - \sum_{i=1}^{n-1} d\omega_i d\omega'_i),$$

where dr means the radial part of the vector $(dr, d\omega) \in T_{(\omega,r)}\mathcal{L}^n$ and $d\omega$ means the part orthogonal, in Euclidean meaning, to the radius. We shall call $d\omega$ the spherical part.

Thus, the Euclidean unit sphere corresponds to the infinity in $\mathcal{L}^n$, the Euclidean 'sphere' of infinite radius corresponds to the equator of $\mathcal{Q}^n_{-1}$. The geodesics are the straightlines, that are timelike, lightlike or spacelike as the straightlines intersect, touch or avoid the unit sphere, respectively. It is worthwhile to note that the timelike and lightlike geodesics are in fact composed from two half straightlines (double-covered).

As usual, we parameterize the set of hyperplanes in $\mathbb{R}^n$, so that $H(\omega, p)$ denotes the hyperplane perpendicular to $\omega \in S^{n-1}$ and going through $p \cdot \omega \in \mathbb{R}^n$, where $p \in \mathbb{R}_+$. The corresponding 1-codimesional totalgeodesic in $\mathcal{L}^n$ will be denoted by $\hat{H}(\omega, p)$. For $p < 1$ this correspondence is not one-to-one in dimension two, because then the intersection of the corresponding plane with $\mathcal{Q}^2_{-1}$ falls into two geodesics. As a matter of fact, this parameterization can not be used in this case generally, therefore we shall use even and odd functions only. Nevertheless, in some cases we can not avoid to use this parameterization for general functions, where it remains the reader's easy task to identify which of the two timelike geodesic is in context.

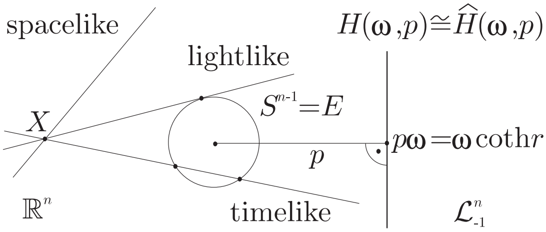

Figure 2.

Because of the rotational symmetry, to calculate the length and angle elements in $\mathcal{L}^n$ we have to work only in dimension two. We use these measures for the Radon transform, but it can be used also for determining the laws of the trigonometry in $\mathcal{L}^n$ for the triangles.

Lemma 2.1. *The arclength measure on the geodesic $\hat{H}(\bar{\omega}, p)$ at the point $X = (\omega, r) \in \mathcal{L}^n$, where $\coth r = p/\langle \omega, \bar{\omega}\rangle$ and $0 < \langle \omega, \bar{\omega}\rangle < p$, is*

$$di = \frac{p\sqrt{|p^2 - 1|}}{p^2 - \langle \omega, \bar{\omega}\rangle^2} d\omega.$$

Proof. If $d\omega$ is the infinitesimal element at ω on S^1, and $\overline{di}$ is the corresponding Euclidean arclength element on $H(\bar{\omega}, p)$ at X, then $\overline{di} = d\omega \cdot p/\langle \bar{\omega}, \omega\rangle^2$. The radial (resp. spherical) part of $\overline{di}$ is $dr = \overline{di} \sin\alpha$ (resp. $ds = \overline{di}\cos\alpha$), where $\cos\alpha = \langle \omega, \bar{\omega}\rangle$. Therefore

$$\begin{aligned} di^2 = \langle \overline{di}, \overline{di}\rangle_X &= \sinh^2 r(dr^2 \sinh^2 r - ds^2) \\ &= \sinh^2 r(\sin^2\alpha \sinh^2 r - \cos^2\alpha) \cdot \frac{p^2 d\omega^2}{\langle \omega, \bar{\omega}\rangle^4}. \end{aligned}$$

Using $\sinh^2 r = (\coth^2 r - 1)^{-1} = \cos^2\alpha/(p^2 - \cos^2\alpha)$ by the definition of $\cos\alpha$, this gives the statement.

■

Although the parameter p seems very artificial in this setting, it can be determined from inside the Lorentz space.

Let X be a point in $\mathcal{L}^n$. We are looking for an angle measure in the tangent space T_X that is invariant under the isotropy group. In other words we want to have a distance on the unit 'sphere' in T_X according to the inner product $\langle ., .\rangle_X$. Let

$$d(\omega_1, \omega_2) = \frac{\langle \omega_1, \omega_2\rangle_X^2}{\langle \omega_1, \omega_1\rangle_X \langle \omega_2, \omega_2\rangle_X}. \tag{2.6}$$

Then d is a function on the pairs of elements of the unit sphere in T_X, and it is invariant under the isotropy group obviously. (We suppose neither ω_1 nor ω_2 is lightlike.)

At the same time one sees that the Lorentzian sphere falls into parts, determined by the timelike and spacelike vectors. These are called timelike and spacelike spheres, respectively. In dimension two there are two connected spacelike spheres and two connected timelike spheres. In higher dimensions there are still two connected timelike spheres, but only one spacelike sphere.

We define the angle of $\omega_1, \omega_2 \in T_X$ for timelike and spacelike vectors differently. Let $L_X \subset T_X$ be the cone of lightlike vectors, C_X be the set of timelike vectors and D_X be the set of spacelike vectors. Then we set the angle γ of ω_1 and ω_2 as

$$\begin{aligned} \cosh^2\gamma &= d(\omega_1,\omega_2) \quad \text{if} \quad \omega_1,\omega_2 \in C_X, \\ \sinh^2\gamma &= -d(\omega_1,\omega_2) \quad \text{if} \quad \omega_1,\omega_2 \in D_X. \end{aligned} \tag{2.7}$$

We denote the Lorentzian unit sphere at $X \in \mathcal{L}^n$ by Σ_X^{n-1}.

Lemma 2.2. *Let $X = (\bar\omega, r) \in \mathcal{L}^2$ and for each $\omega \in S^1$ let $\hat\omega \in \Sigma_X^1$ be tangent to $\hat H(\omega, \coth r\langle\omega,\bar\omega\rangle)$. Then*

$$d\hat\omega = \frac{1/|\sinh r|}{\coth^2 r\langle\omega,\bar\omega\rangle^2 - 1}\,d\omega \quad \textit{on} \quad \Sigma_X^1 \setminus L_X.$$

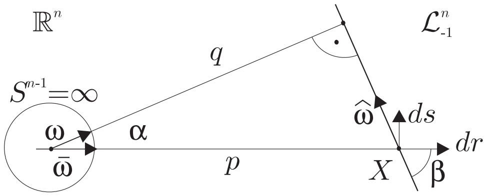

Figure 3.

Proof. (See Figure 3.) Let dr and ds denote the radial and spherical part of $\hat\omega$, respectively. Then $\hat\omega = dr\cos\beta + ds\sin\beta$ for suitable β, and simple calculation gives that

$$d(\hat\omega, dr) = \frac{\cos^2\beta}{1 - \coth^2 r\sin^2\beta}. \tag{2.8}$$

If $\cos\alpha = \langle\omega,\bar\omega\rangle$ we see that $\alpha+\beta = \pi/2$. Let γ be the Lorentzian angle of $\hat\omega$ and dr as defined by (2.7). For timelike $\hat\omega$ we have $\cosh^2\gamma = d(\hat\omega, dr)$. Then differentiating (2.8) with respect to α we obtain

$$d\gamma = \frac{-1/|\sinh r|}{1 - \coth^2 r\cos^2\alpha}\,d\alpha.$$

The formula for $\hat\omega \in D_X$ can be proved in the same way.

■

Since S^{n-1} and Σ_X^{n-1} are both rotational symmetric with respect to the axis OX the surface measure on the Lorentzian unit sphere Σ_X^{n-1} around X with respect to the Euclidean measure on S^{n-1} is given with the same formula as that of Lemma 2.2.

Now, we have all the necessities about the Lorentzian to calculate with the Lorentzian Radon transform.

3. The Radon transform and its dual

In this section we determine explicitly the Radon transform and its dual. The Radon transform integrates a 'good enough' function f on the 1-codimensional totally geodesic submanifolds, that is

$$Rf(\omega,p) = Rf(\hat H(\omega,p)) = \int_{\hat H(\omega,p)} f(X)dX, \tag{3.1}$$

where dX is the Lorentzian surface measure on $\hat H(\omega,p)$.

To define the dual transform R^t we need a function F, say, continuous on the set of 1-codimensional totally geodesic submanifolds of $\mathcal{L}^n$. Then

$$R^tF(X) = \int_{X\in\hat H;\hat\omega\in\Sigma_X^{n-1}} F(\hat H)d\hat\omega, \tag{3.2}$$

where $d\hat\omega$ is the Lorentzian measure on Σ_X^{n-1} and $\hat\omega$ is the normal of $\hat H$ at $X \in \mathcal{L}^n$, that is $d(\hat\omega,\eta) = 0$ for any $\eta \in T_X\hat H$.

Proposition 3.1. (i) *The Radon transform of $f \in S(\mathcal{L}^2)$ is*

$$Rf(\omega_{\bar\varphi},p) = \int_{S_p^1} f\Big(\omega_{\bar\varphi+\varphi}, \operatorname{arccoth}\Big(\frac{p}{\cos\varphi}\Big)\Big)\frac{p\sqrt{|p^2-1|}}{p^2-\cos^2\varphi}d\varphi, \tag{3.3}$$

where $\bar\varphi$ and $\varphi+\bar\varphi$ are the angles of the unit vectors $\omega_{\bar\varphi}$ and $\omega_{\bar\varphi+\varphi}$ to a fixed direction respectively, and

$$S_p^1 = \begin{cases} [-\pi,\pi] & \text{if } p>1 \\ ((-\pi/2,0]\cup(\pi/2,\pi])\cap\{\varphi : |\cos\varphi|<p\} & \text{if } p<1. \end{cases}$$

(ii) *The Radon transform of $f \in S(\mathcal{L}^n)$ $(n > 2)$ is*

$$Rf(\bar\omega, p) = \int_{S^{n-1}_{\bar\omega,p}} f\Big(\omega, \operatorname{arccoth}\Big(\frac{p}{\langle\omega,\bar\omega\rangle}\Big)\Big) \frac{p^{n-1}\sqrt{|p^2-1|}}{(p^2-\langle\omega,\bar\omega\rangle^2)^{n/2}} d\omega, \tag{3.4}$$

where $S^{n-1}_{\bar\omega,p} = \{\omega \in S^{n-1} : 0 < \langle\omega,\bar\omega\rangle < p\}$.

Note that the difference in the definition of S^1_p for $p > 1$ and $p < 1$ appears according to the type of the geodesics the integration is taken along. Note further, that f is parameterized by the natural metric on $\mathcal{L}^n$ contrary to the function Rf which is parameterized according to the correspondence between $H(\bar\omega, p)$ and $\hat H(\bar\omega, p)$.

Proof. We have to determine the surface element dh of $\hat H(\bar\omega, p)$ with respect to $d\omega$ at the point $p\omega/\langle\omega,\bar\omega\rangle$ given in Euclidean coordinates. In dimension two, Lemma 2.1 gives $dh = di$.

For higher dimensions, first we observe, that the sphere $\mu(R)$ of radius $\coth r$ centered to the origin in $\mathbb{R}^n$ corresponds to the 'ring' $R = \mathcal{Q}^n_{-1} \cap \{x_1 = \sinh r\}$ in $\mathcal{Q}^n_{-1}$ that has radius $\cosh r$ in $\mathbb{R}^{n+1}$. By (2.1) this means that the natural map $\nu\colon S^{n-1} \to \mu(R)$ induces a dilation ν^* of coefficient $\cosh r$ between the corresponding tangent spaces of S^{n-1} and $\mu(R)$. This and the rotational symmetry implies that

$$dh = \frac{p\sqrt{|p^2-1|}}{p^2-\langle\omega,\bar\omega\rangle^2} \cdot \cosh^{n-2}\Big(\operatorname{arccoth}\Big(\frac{p}{\langle\omega,\bar\omega\rangle}\Big)\Big) d\omega,$$

that gives the proposition immediately.

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Note that the Radon transform of any function on the lightlike totally geodesics ($p = 1$) is always zero. To calculate the dual transform we introduce $\mathcal{H}^n$ as the manifold of the 1-codimensional totally geodesic submanifolds in $\mathcal{L}^n$.

Proposition 3.2. (i) *The dual Radon transform of $F \in L^2(\mathcal{H}^2)$ is*

$$R^tF(X) = R^tF(\omega_{\bar\varphi}, r) = \int_{-\pi/2}^{\pi/2} F(\omega_{\bar\varphi+\varphi}, \coth r \cos\varphi) \frac{1/|\sinh r|}{\coth^2 r \cos^2\varphi - 1} d\varphi. \tag{3.5}$$

(ii) *The dual Radon transform of $F \in L^2(\mathcal{H}^n)$, $n > 2$, is*

(3.6)

$$R^tF(X) = R^tF(\bar\omega, r) = \int_{S^{n-1}, \langle\omega,\bar\omega\rangle>0} F(\omega, \coth r\langle\omega,\bar\omega\rangle) \frac{1/|\sinh r|}{|\coth^2 r\langle\omega,\bar\omega\rangle^2 - 1|} d\omega$$

where $X = (\bar\omega, r) \in \mathcal{L}^n$ *and* $F(\omega, p)$ *is a shorthand for* $F(\hat H(\omega, p))$.

Notice that F is parameterized according to the correspondence between $H(\bar\omega, p)$ and $\hat H(\bar\omega, p)$ while the function $R^t F$, as well as X, is parameterized by the natural metric on $\mathcal{L}^n$.

Proof. The measure comes from Lemma 2.2, while the parameters of F are implied by elementary geometrical calculations.

■

We define the spacelike Radon transform R_S and timelike Radon transform R_T, respectively, of a function $f \in L^2(\mathcal{L}^n)$ so by

$$Rf(\omega, p) = \left\{ \begin{array}{ll} R_S f(\omega, p) & \text{if } \ p > 1 \\ R_T f(\omega, p) & \text{if } \ p < 1 \end{array} \right\}. \tag{3.7}$$

The spacelike Radon transform integrates over the compact totally geodesics, and the timelike Radon transform integrates over the non-compact totally geodesics. In dimension two these are the spacelike (compact) and the timelike (non-compact) geodesics.

The spacelike and timelike dual Radon transform in dimension $n > 2$ is defined by

$$\begin{aligned} R^t_S F(\bar\omega, r) &= \int_{1 < \langle \omega, \bar\omega\rangle \coth r} F(\omega, \coth r \langle \omega, \bar\omega\rangle) \frac{1/|\sinh r|}{\coth^2 r \langle \omega, \bar\omega\rangle^2 - 1} d\omega, \\ R^t_T F(\bar\omega, r) &= \int_{0 < \langle \omega, \bar\omega\rangle \coth r < 1} F(\omega, \coth r \langle \omega, \bar\omega\rangle) \frac{1/|\sinh r|}{1 - \coth^2 r \langle \omega, \bar\omega\rangle^2} d\omega. \end{aligned} \tag{3.8}$$

For R^t we have $R^t = R^t_S + R^t_T$. In dimension two the corresponding definitions are

$$\begin{aligned} R^t_S F(\omega_{\bar\varphi}, r) &= \int_{1 < \cos\varphi \coth r} F(\omega_{\bar\varphi + \varphi}, \coth r \cos\varphi) \frac{1/|\sinh r|}{\coth^2 r \cos^2\varphi - 1} d\varphi, \\ R^t_T F(\omega_{\bar\varphi}, r) &= \int_{\coth r |\cos\varphi| < 1} F(\omega_{\bar\varphi + \varphi}, \coth r \cos\varphi) \frac{1/|\sinh r|}{1 - \coth^2 r \cos^2\varphi} d\varphi. \end{aligned} \tag{3.9}$$

Let f be a function on $\mathcal{Q}^n_{-1}$ integrable on each 1-codimensional totalgeodesic. We say $f_e(x) = (f(x) + f(-x))/2$ is the even and $f_o(x) = (f(x) - f(-x))/2$ is the odd part of f, respectively. A function is called even resp. odd if $f_o = 0$ resp. $f_e = 0$. In the model $\mathcal{L}^n$ a function f is even if $f(\omega, p) = f(-\omega, -p)$ and is odd if $f(\omega, p) = -f(-\omega, -p)$.

In dimensions higher than two, both the spacelike and the timelike Radon transform of the odd part is zero, therefore we shall consider only the even functions in these spaces.

In dimension two the spacelike Radon transform of an odd function is still the zero but the timelike Radon transform becomes non trivial. To explore this situation we shall investigate the two dimensional case in the last section separately.

Now we are giving the spherical harmonic expansions of our transforms. Doing this, we calculate with the even and the odd functions separately that greatly simplifies our calculations, because then we can avoid complicated representation of the double covering and simply calculate with functions defined on $\mathbb{E}^n = \mathbb{R}^n \setminus B^n$, where B^n is the open unit ball in $\mathbb{R}^n$.

Briefly, the spherical harmonics, $Y_{\ell,m}$ constitute a complete polynomial orthonormal system in the Hilbert space $L^2(S^{n-1})$. If $f \in C^\infty(S^{n-1} \times \mathbb{R}_+)$ and $f_{\ell,m}(p)$ is the corresponding coefficient of $Y_{\ell,m}(\omega)$ in the expansion of $f(\omega,p)$, ie. $f_{\ell,m}(p) = \int_{S^{n-1}} f(\omega,p)\overline{Y_{\ell,m}(\omega)}d\omega$, then the series $\sum_{\ell,m}^\infty f_{\ell,m}(p)Y_{\ell,m}(\omega)$ converges uniformly absolutely on compact subsets of $S^{n-1} \times \mathbb{R}$ to $f(\omega,p)$. For further references, including the Funk-Hecke theorem,

$$Y_{\ell,m}(\bar\omega)\frac{|S^{n-2}|}{C_m^\lambda(1)}\int_{-1}^1 C_m^\lambda(x)(1-x^2)^{\lambda-\frac12}f(x)dx = \int_{S^{n-1}} f(\langle\omega,\bar\omega\rangle)Y_{\ell,m}(\omega)d\omega, \tag{3.10}$$

where C_m^λ is the Gegenbauer polynomial of degree m and $\lambda = (n-2)/2$, we refer to [10]. Below we shall use the expansions

$$g(\omega_\varphi,p) = \sum_{m=-\infty}^{\infty} g_m(p)\exp(im\varphi) \quad \text{and} \quad g(\omega,p) = \sum_{\ell,m}^{\infty} g_{\ell,m}(p)Y_{\ell,m}(\omega) \tag{3.11}$$

for dimension two and for higher dimensions, respectively. Though a little confusing, in the followings we allow the shorthand notation φ for ω_φ for simplifying the notation.

Proposition 3.3. (i) *If $f(\varphi,p) \in L^2(\mathcal{L}^2)$ is even, then*

(3.12)

$$(R_S f)_m(p) = 4\int_0^{\operatorname{arccoth} p} f_m(q)\frac{\cos\left(m\arccos\left(\frac{p}{\coth q}\right)\right)\sqrt{p^2-1}}{\sqrt{1-p^2/\coth^2 q}}dq, \qquad (p>1)$$

$$(R_T f)_m(p) = 2\int_0^{\infty} f_m(q)\frac{\cos\left(m\arccos\left(\frac{p}{\coth q}\right)\right)\sqrt{1-p^2}}{\sqrt{1-p^2/\coth^2 q}}dq. \qquad (p<1)$$

(ii) *If* $f(\varphi,p)\in L^2(\mathcal{L}^2)$ *is odd then*

$$(R_Tf)_m(p)=2i\int_0^\infty f_m(q)\frac{\sqrt{1-p^2}\sin\Big(m\arccos\Big(\frac{p}{\coth q}\Big)\Big)}{\sqrt{1-p^2/\coth^2 q}}dq\qquad (p<1). \tag{3.13}$$

Proof. For (i) let $f(\varphi,p)\in L^2(\mathcal{L}^2)$ be even. Then (3.3) can be read as

$$\begin{aligned}R_Sf(\bar\varphi,p)&=2\int_{-\pi/2}^{\pi/2} f\Big(\bar\varphi+\varphi,\operatorname{arccoth}\Big(\frac{p}{\cos\varphi}\Big)\Big)\frac{p\sqrt{p^2-1}}{p^2-\cos^2\varphi}d\varphi,\\ R_Tf(\bar\varphi,p)&=\int\limits_{-\pi/2}^{-\arccos p}+\int\limits_{\arccos p}^{\pi/2} f\Big(\bar\varphi+\varphi,\operatorname{arccoth}\Big(\frac{p}{\cos\varphi}\Big)\Big)\frac{p\sqrt{1-p^2}}{p^2-\cos^2\varphi}d\varphi.\end{aligned}\tag{3.14}$$

Substituting the (3.11)-type expansions of Rf and f into these formulas we get

$$\begin{aligned}(R_Sf)_m(p)&=2\int\limits_{-\pi/2}^{\pi/2} f_m\Big(\operatorname{arccoth}\Big(\frac{p}{\cos\varphi}\Big)\Big)\frac{\exp(im\varphi)p\sqrt{p^2-1}}{p^2-\cos^2\varphi}d\varphi,\\ (R_Tf)_m(p)&=\int\limits_{-\pi/2}^{-\arccos p}+\int\limits_{\arccos p}^{\pi/2} f_m\Big(\operatorname{arccoth}\Big(\frac{p}{\cos\varphi}\Big)\Big)\frac{\exp(im\varphi)p\sqrt{1-p^2}}{p^2-\cos^2\varphi}d\varphi.\end{aligned}\tag{3.15}$$

Since the domains of these integrations are symmetric to the origin the imaginary part of the factor $\exp(im\varphi)$ makes zero in the integral. Therefore the substitution $\cos\varphi=p/\coth q$ gives the desired formulas.

To see (ii) let $f(\varphi,p)\in L^2(\mathcal{L}^2)$ be odd. Then the second part of (3.15) changes to

$$(R_Tf)(\bar\varphi,p)=\int\limits_{-\pi/2}^{-\arccos p}-\int\limits_{\arccos p}^{\pi/2} f\Big(\bar\varphi+\varphi,\operatorname{arccoth}\Big(\frac{p}{\cos\varphi}\Big)\Big)\frac{p\sqrt{1-p^2}}{p^2-\cos^2\varphi}d\varphi$$

which gives

$$(R_Tf)_m(p)=\int\limits_{-\pi/2}^{-\arccos p}-\int\limits_{\arccos p}^{\pi/2} f_m\Big(\operatorname{arccoth}\Big(\frac{p}{\cos\varphi}\Big)\Big)\frac{\exp(im\varphi)p\sqrt{1-p^2}}{p^2-\cos^2\varphi}d\varphi.$$

Now the real part of the factor $\exp(im\varphi)$ makes zero in the integral hence again the substitution $\cos\varphi=p/\coth q$ leads to (ii).

■

For higher dimensions we have the following

Proposition 3.4. *If $f(\omega,p)\in L^2(\mathcal{L}^n)$ is even and $n>2$ then*

$$(3.16)\qquad \begin{aligned}(Rf)_{l,m}(p)=\frac{2|S^{n-2}|}{C_m^\lambda(1)}\int_0^{v(p)} f_{\ell,m}(q)\sqrt{|p^2-1|}\cosh^{n-2}q\times\\ \times\Big(1-\frac{p^2}{\coth^2 q}\Big)^{\frac{n-3}{2}}C_m^\lambda\Big(\frac{p}{\coth q}\Big)dq,\end{aligned}$$

where $v(p)=\begin{cases}\infty & \text{if } p\le 1,\\ \operatorname{arccoth} p & \text{if } p>1.\end{cases}$

Proof. Because f is even (3.4) reads

$$(3.17)\qquad Rf(\bar\omega,p)=2\int_{0<\langle\omega,\bar\omega\rangle<p} f\Big(\omega,\operatorname{arccoth}\Big(\frac{p}{\langle\omega,\bar\omega\rangle}\Big)\Big)\frac{p^{n-1}\sqrt{|p^2-1|}}{(p^2-\langle\bar\omega,\omega\rangle^2)^{n/2}}d\omega.$$

The Funk-Hecke theorem (3.10) implies

$$\begin{aligned}\int\limits_{0<\langle\omega,\bar\omega\rangle<p} f_{\ell,m}\Big(\operatorname{arccoth}\Big(\frac{p}{\langle\omega,\bar\omega\rangle}\Big)\Big)\frac{Y_{\ell,m}(\omega)p^{n-1}\sqrt{|p^2-1|}}{(p^2-\langle\omega,\bar\omega\rangle^2)^{\pi/2}}d\omega\\ =Y_{\ell,m}(\bar\omega)\frac{|S^{n-2}|}{C_m^\lambda(1)}\int\limits_0^{\min(p,1)} f_{\ell,m}\Big(\operatorname{arccoth}\Big(\frac{p}{t}\Big)\Big)\frac{p^{n-1}\sqrt{|p^2-1|}}{(p^2-t^2)^{n/2}}\times\\ \times(1-t^2)^{\lambda-1/2}C_m^\lambda(t)dt\end{aligned}$$

hence

$$\begin{aligned}(Rf)_{\ell,m}(p)=\frac{2|S^{n-2}|}{C_m^\lambda(1)}\int\limits_0^{\min(p,1)} f_{\ell,m}\Big(\operatorname{arccoth}\Big(\frac{p}{t}\Big)\Big)\frac{p^{n-1}\sqrt{|p^2-1|}}{(p^2-t^2)^{n/2}}\times\\ \times(1-t^2)^{\lambda-1/2}C_m^\lambda(t)dt.\end{aligned}$$

Substituting $t=p/\coth q$ leads to the desired formula. ∎

Because we shall not need it in this paper, we present the following result without details.

Proposition 3.5. (i) *If $F(\varphi,p)\in L^2(\mathcal{H}^2)$ then*

$$(R_S^tF)_m(r)=2\int_1^{\coth r}F_m(q)\frac{1/\cosh r}{q^2-1}\frac{\cos(m\arccos\left(\frac{q}{\coth r}\right))}{\sqrt{1-q^2/\coth^2 r}}dq.$$

(ii) *If $F(\omega,p)\in L^2(\mathcal{H}^n)$, $n>2$, then*

$$(R_S^tF)_{\ell,m}(r)=\frac{|S^{n-2}|}{C_m^\lambda(1)}\int\limits_1^{\coth r}F_{\ell,m}(q)\frac{1/\coth r}{q^2-1}C_m^\lambda\left(\frac{q}{\coth r}\right)\left(1-\frac{q^2}{\coth^2 r}\right)^{\lambda-1/2}dq$$

and

$$(R_T^tF)_{\ell,m}(r)=\frac{|S^{n-2}|}{C_m^\lambda(1)}\int_0^1F_{\ell,m}(q)\frac{1/\cosh r}{q^2-1}C_m^\lambda\left(\frac{q}{\coth r}\right)\left(1-\frac{q^2}{\coth^2 r}\right)^{\lambda-1/2}dq.$$

4. Even functions in higher dimensions

As we already noted the odd part of any function is annihilated by R in dimensions higher than two and so only the even part can be recovered. Therefore in this section we investigate only even functions. Considering only even functions allows us to use a simplified model without double-covering.

The same simplification can be seen at the dual Radon transform in higher dimensions, because then all the 1-codimensional totally geodesic submanifolds of $\mathcal{Q}^n_{-1}$ are symmetric to the origin of $\mathbb{R}^{n+1}$.

Theorem 4.1. *Let $n>2$ and $g\in L^2(\mathbb{E}^n)$. If $f(\omega,r)=g(\omega,\coth r)(\coth^2 r-1)^{n/2}$ then*

$$\bar{R}g(\bar{\omega},p)=Rf(\bar{\omega},p)\cdot\frac{1/2}{\sqrt{|p^2-1|}}\qquad(p\neq 1),$$

where $\bar{R}$ denotes the Euclidean Radon transform on $\mathbb{R}^n$.

Before the proof we note that f is well defined as an even function on $\mathcal{L}^n$.

Proof. It is well known [5] that

$$2\cdot \bar{R}g(\bar\omega,p)=\int_{S^{n-1}} g\Big(\omega\cdot\frac{p}{\langle\omega,\bar\omega\rangle}\Big)\frac{p^{n-1}}{|\langle\omega,\bar\omega\rangle|^{n}}d\omega.$$

On the other hand, by Proposition 3.1 we have

$$\begin{aligned}\frac{Rf(\bar\omega,p)}{2\sqrt{|p^2-1|}}&=\frac12\int_{S^{n-1}_{\bar\omega,p}} f\Big(\omega,\operatorname{arccoth}\Big(\frac{p}{\langle\omega,\bar\omega\rangle}\Big)\Big)\frac{p^{n-1}}{(p^2-\langle\omega,\bar\omega\rangle^2)^{n/2}}d\omega\\&=\frac12\int_{S^{n-1}_{\bar\omega,p}} g\Big(\omega\frac{p}{\langle\omega,\bar\omega\rangle}\Big)\Big(\frac{p^2}{\langle\omega,\bar\omega\rangle^2}-1\Big)^{n/2}\frac{p^{n-1}}{(p^2-\langle\omega,\bar\omega\rangle^2)^{n/2}}d\omega\end{aligned}$$

that gives the statement immediately. ∎

It is worthwhile to note that if $g\in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp} g\subseteq\mathbb{E}^n$ then $f\in S(\mathcal{L}^n\setminus E)$, where E is the equator. Furthermore, if $g\in S(\mathbb{E}^n)$ then $f\in S(\mathcal{L}^n)$ and f and all of its derivatives are zero at the equator.

Theorem 4.1 can be used to transfer most of the results known on the Euclidean space to the Lorentzian space. Here, we make the transfer only for the support theorem and the inversion formula.

Since all the geodesics meet the equator, the analog of the Euclidean support theorem is the following. (Note that only the spacelike totalgeodesics are near the equator.)

Theorem 4.2. *If $f\in C(\mathcal{L}^n)$ is even, f with all of its derivatives are zero at the equator E and Rf is zero in a belt B around E then f is zero in B.*

Proof. Let $g\colon\mathbb{E}^n\to\mathbb{R}$ be defined by

$$g(\omega,\coth r)=f(\omega,r)\cdot(\coth^2 r-1)^{-n/2}.$$

Then $\bar{R}g(\bar\omega,p)=Rf(\bar\omega,p)\frac{1/2}{\sqrt{|p^2-1|}}$. Our condition says $Rf(\bar\omega,p)=0$ if $p>A$ for some $1<A<\infty$. Therefore we only have to show that g satisfies the conditions of Helgason's support theorems [5]. One can obviously alter g near the unit sphere so that it becomes continuous on $\mathbb{R}^n$ hence only $\lim_{x\to\infty}g(x)|x|^m=0$ is needed to be shown for all $m\in\mathbb{N}$. This is proved by the following sequence of equations.

$$\begin{aligned}\lim_{x\to\infty}g(x)|x|^m&=\lim_{r\to0}g(\bar\omega\coth r)(\coth^2 r-1)^{-n/2}\coth^{2m} r\\&=\lim_{r\to0}f(\bar\omega,r)\sinh^{n-2m}r=\lim_{r\to0}\frac{\frac{\partial^{2m-n}}{\partial r^{2m-n}}f(\bar\omega,r)}{(2m-n)!}=0.\end{aligned}$$ ∎

■

By the homogeneity of $\mathcal{L}^n$ Theorem 4.2 is valid for any spacelike totalgeodesic in place of the equator. To formulate the inversion formula we define the operator Λ by

$$\Lambda\Phi(\omega,p) = \begin{cases} \dfrac{\partial^{n-1}}{\partial p^{n-1}}\Phi(\omega,p) & n \text{ odd} \\ \dfrac{i}{\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{\partial^{n-1}}{\partial t^{n-1}}\Phi(\omega,p)\dfrac{dt}{t-p} & n \text{ even} \end{cases}$$

for Φ in the Schwartz space of the functions on the set of hyperplanes in $\mathbb{R}^n$.

Theorem 4.3. *For $f \in S(\mathcal{L}^n)$ even that is zero with all of its derivatives at the equator*

$$cf = \sinh^{1-n} r R^t\left(\frac{1}{|1-p^2|}\Lambda\left(\frac{Rf(\omega,p)}{2\sqrt{|1-p^2|}}\right)\right),$$

where $c = (-4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)$.

Proof. Observe $|\sinh r|R^t F(\omega,r) = \bar{R}^* G(\omega,\coth r)$ for $G(\omega,p) = F(\omega,p)/|1-p^2|$, where $\bar{R}^*$ is the Euclidean dual Radon transform. Substitute this and Theorem 4.1 into Helgason's Theorem 3.4 in [5].

■

This result shows that in odd dimensions the inversion is local contrary the even dimensions, where the reconstruction needs Rf on all the total geodesics. On the other hand, this inversion formula hides an important aspect of the Lorentzian Radon transform, namely, that the spacelike and timelike, for even dimensions only, Radon transform is injection.

We shall need the formula

(4.1)

$$M\left(\frac{\sinh(s-q)}{\cosh q\cosh s}\right)^{n-2} = \int_q^s C_m^\lambda\left(\frac{\coth r}{\coth q}\right)\left(1-\frac{\coth^2 r}{\coth^2 q}\right)^{\frac{n-3}{2}} \times$$
$$\times C_m^\lambda\left(\frac{\coth r}{\coth s}\right)\left(\frac{\coth^2 r}{\coth^2 s}-1\right)^{\frac{n-3}{2}} \cdot \frac{\tanh^{n-3} r}{\cosh^2 r}dr,$$

where

$$M = \frac{\pi 2^{3-n}}{\Gamma(n-1)}\left(\frac{\Gamma(m+n-2)}{\Gamma(m+1)\Gamma(\lambda)}\right)^2.$$

It can be proven by a simple substitution from the corresponding formula of [1]. The following result gives an inversion formula for the spacelike Radon transform in terms of spherical harmonic expansions.

Theorem 4.4. *If $f \in C^\infty(\mathcal{L}^n)$ $(n > 2)$ is even and zero in a neighborhood of the equator E then*

$$f_{\ell,m}(s) = (-1)^{n-1}\frac{\Gamma(m+1)\Gamma(\lambda)}{\pi^{n/2}\Gamma(m+n-2)}\begin{cases}\frac{d}{ds}\delta_2\delta_4\cdots\delta_{n-2}F_{\ell,m}(s) & \textit{if } n \textit{ even}\\ \delta_1\delta_3\delta_5\ldots\delta_{n-2}F_{\ell,m}(s) & \textit{if } n \textit{ odd,}\end{cases}$$

where $\delta_k = \frac{d^2}{ds^2} - k^2$ $(k \in \mathbb{N})$, $\lambda = (n-2)/2$ *and*

$$F_{\ell,m}(s) = -\int_{\coth s}^{\infty}(R_S f)_{\ell,m}(p)C_m^\lambda\left(\frac{p}{\coth s}\right)\left(\frac{p^2}{\coth^2 s}-1\right)^{\frac{n-3}{2}}\frac{\cosh^{n-2}s}{p^{n-1}\sqrt{p^2-1}}dp.$$

Proof. First we multiply the formula of Proposition 3.4 with

$$-C_m^\lambda\left(\frac{p}{\coth s}\right)\left(\frac{p^2}{\coth^2 s}-1\right)^{\frac{n-3}{2}}\frac{\cosh^{n-2}s}{p^{n-1}\sqrt{p^2-1}}$$

and integrate from $\coth s$ to ∞ by p. Denoting the result by $F_{\ell,m}$ we get

$$\begin{aligned}F_{\ell,m}(s) = -\int_{\coth s}^{\infty} C_m^\lambda\left(\frac{p}{\coth s}\right)\left(\frac{p^2}{\coth^2 s}-1\right)^{\frac{n-3}{2}}\frac{\cosh^{n-2}s}{p^{n-1}\sqrt{p^2-1}}\times\\ \times\frac{2|S^{n-2}|}{C_m^\lambda(1)}\int_0^{\operatorname{arccoth} p} f_{\ell,m}(q)C_m^\lambda\left(\frac{p}{\coth q}\right)\left(1-\frac{p^2}{\coth^2 q}\right)^{\frac{n-3}{2}}\times\\ \times\cosh^{n-2}q\sqrt{p^2-1}\,dq\,dp.\end{aligned}$$

Substituting the variable $p = \coth r$ and then changing the order of the integrations we see

(4.2)

$$\begin{aligned}F_{\ell,m}(s) = \frac{2|S^{n-2}|}{C_m^\lambda(1)}\int_0^s f_{\ell,m}(q)(\cosh s\cosh q)^{n-2}\times\\ \times\int_q^s C_m^\lambda\left(\frac{\coth r}{\coth s}\right)C_m^\lambda\left(\frac{\coth r}{\coth q}\right)\left(\frac{\coth^2 r}{\coth^2 s}-1\right)^{\frac{n-3}{2}}\times\\ \times\left(1-\frac{\coth^2 r}{\coth^2 q}\right)^{\frac{n-3}{2}}\frac{\tanh^{n-1}r}{\sinh^2 r}dr\,dq.\end{aligned}$$

According to formula 4.1 we obtain

$$F_{\ell,m}(s) = \frac{2M|S^{n-2}|}{C_m^\lambda(1)}\int_0^s f_{\ell,m}(q)\sinh^{n-2}(s-q)dq$$

which implies the formula by the observation

$$\frac{d^2}{ds^2}\sinh^k(s-q) = k^2\sinh^k(s-q) + k(k-1)\sinh^{k-2}(s-q). \qquad \blacksquare$$

The connection between the spacelike Radon transform and the exterior Radon transform [9] is striking. A singular value decomposition can be easily calculated from Quinto's result via Theorem 4.1.

For the timelike Radon transform the integration in the formula of Proposition 3.4, is taken from zero to infinity hence in odd dimensions $(R_T f)_{\ell,m}$ determines only some moments of $f_{\ell,m}$. Therefore the timelike Radon transform is not invertible in odd dimensions. Nevertheless one could look for its null space and range, but having Theorem 4.4 we leave the details for the interested reader and give only the result for even dimensions.

Theorem 4.5. *If $f \in L^2(\mathcal{L}^n)$, $n > 2$ even and the timelike Radon transform of f is zero then f is odd.*

For details we refer to [7], where analogous results are provided for the Euclidean spaces.

5. The dimension is two

Beside of the other mainly technical differences we must consider the two dimensional case separately because the timelike Radon transform does not vanish on the space of the odd function. First we consider the even functions. The proof of the following theorem is identical with that of Theorem 4.1.

Theorem 5.1. *Let $g \in L^2(\mathbb{E}^2)$. Define $f(\omega, r) = g(\omega, \coth r)(\coth^2 r - 1)$ as even function on $\mathcal{L}^2$. Then*

$$\bar{R}g(\bar{\omega}, p) = \begin{cases} Rf(\bar{\omega}, p)/\sqrt{1-p^2} & p < 1 \\ Rf(\bar{\omega}, p)/(2\sqrt{p^2-1}) & 1 < p, \end{cases}$$

where $\bar{R}$ denotes the Euclidean Radon transform on $\mathbb{R}^2$.

Theorem 4.2 remains valid without modification (of course one has to allow $n = 2$). Theorem 4.3 is also valid for dimension two. Theorem 4.4 has to be modified, but remains valid in its spirit.

Theorem 5.2. *If $f \in C^\infty(\mathcal{L}^2)$ is even and zero in a neighborhood of the equator E then*

$$f_m(s) = \frac{-1}{2\pi}\frac{d}{ds}\int_{\coth s}^{\infty} (R_S f)_m(p) \frac{\cosh(m \operatorname{arccosh}(p/\coth s))}{p\sqrt{p^2-1}\sqrt{p^2/\coth^2 s - 1}} dp.$$

Proof. Multiply the first formula of Proposition 3.3 with

$$\frac{\cosh(m \operatorname{arccosh}(p/\coth s))}{p\sqrt{p^2-1}\sqrt{p^2/\coth^2 s - 1}}$$

and integrate from $\coth s$ to ∞ by p. The result is

$$F_m(s) = -\int_{\coth s}^{\infty} \frac{\cosh(m \operatorname{arccosh}(p/\coth s))}{p\sqrt{p^2/\coth^2 s - 1}} \times$$
$$\times 4 \int_0^{\operatorname{arccoth} p} f_m(q) \frac{\cos(m \arccos(p/\coth q))}{\sqrt{1 - p^2/\coth^2 q}} dq dp.$$

Substituting $p = \coth r$ and changing the order of the integrations lead to (5.1)

$$F_m(s) = -4\int_0^s f_m(q) \int_q^s \frac{\cos(m \arccos(\coth r/\coth q))}{\sqrt{1 - \coth^2 r/\coth^2 q}} \times$$
$$\times \frac{\cosh(m \operatorname{arccosh}(\coth r/\coth s))}{\sqrt{\coth^2 r/\coth^2 s - 1}} \frac{\tanh r}{\sinh^2 r} dr dq.$$

The inner integral is known to be $\pi/2$ by [1]. Thus we have

$$F_m(s) = 2\pi \int_0^s f_m(q) dq$$

that gives the statement. ■

Now, we are interested in what extend the odd functions can be reconstructed from the timelike Radon transform.

Theorem 5.3. *The null space of R_T in $L^2_{\mathrm{odd}}(\mathcal{L}^2)$ is the closure of the span of functions*

$$f_{\ell,m}(\varphi, r) = \cosh r P_\ell^{(-1/2,\varepsilon_m)}(2\tanh^2 r - 1)\sin(m\varphi),$$

where $P_\ell^{(\alpha,\beta)}$ are the shifted Jacobi polynomials [2] *of order ℓ,*

$$\ell \ge \left[\frac{m+1}{2}\right] \quad \text{and} \quad \varepsilon_m = \begin{cases} 0 & \text{if } m \text{ is even} \\ -1/2 & \text{if } m \text{ is odd.} \end{cases}$$

Proof. We start with (3.13) and observe that the Chebyshev polynomials of second kind [2] are of the form

$$U_{m-1}(x)=\frac{\sin(m\arccos(x))}{\sqrt{1-x^2}}=\sum_{i=0}^{[(m-1)/2]}c_{m,m-1-2i}x^{m-1-2i}.$$

where $c_{m,k}\neq 0$ by [2(8.941)]. This implies that R_T is a polynomial of form

$$\frac{(R_Tf)_m(p)}{2i\sqrt{1-p^2}}=\sum_{i=0}^{[(m-1)/2]}c_{m,m-1-2i}p^{m-1-2i}\int_0^\infty f_m(q)\tanh^{m-1-2i}q\,dq.$$

Therefore $(R_Tf)_m\equiv 0$ if and only if

$$0=\int_0^\infty f_m(q)\tanh^{m-1-2i}q\,dq \quad \text{for all} \quad i\leq\frac{m-1}{2}. \tag{5.2}$$

Let the function ϕ_m defined on $[0,1]$ by $\phi_m(r^2)=f_m(\operatorname{arctanh} r)\sqrt{1-r^2}$. Changing the variable $q=\operatorname{arctanh}\sqrt{x}$ in (5.2) we obtain the equivalent condition that

$$0=\int_0^1\phi_m(x)x^{\frac{m-1}{2}-i}\frac{dx}{\sqrt{x}\sqrt{1-x}} \quad \text{for all} \quad i\leq\frac{m-1}{2}. \tag{5.3}$$

If m is even, (5.3) is equivalent to

$$0=\int_0^1\phi_m(x)x^j\frac{dx}{\sqrt{1-x}} \quad \text{for all} \quad j\leq\frac{m-2}{2}. \tag{5.4}$$

From [2(8.904)] we know that the Jacobi polynomials $P_n^{(-1/2,0)}(2x-1)$ constitute a complete orthogonal system on the interval $[0,1]$ with respect to the weight $1/\sqrt{1-x}$. Therefore ϕ_m should be in the closure of the span of the set $\{P_n^{(-1/2,0)}(2x-1)\}_{n=m/2}^\infty$.

If m is odd, (5.3) is equivalent to

$$0=\int_0^1\phi_m(x)x^j\frac{dx}{\sqrt{x}\sqrt{1-x}} \quad \text{for all} \quad j\leq\frac{m-1}{2}. \tag{5.5}$$

Again [2(8.904)] says that the Jacobi polynomials $P_n^{(-1/2,-1/2)}(2x-1)$ constitute a complete orthogonal system on the interval $[0,1]$ with respect to the weight $1/(\sqrt{x}\sqrt{1-x})$. Therefore ϕ_m should be in the closure of the span of the set $\{P_n^{(-1/2,-1/2)}(2x-1)\}_{n=(m+1)/2}^\infty$.

Summing up the results of (5.4) and (5.5) we obtain the statement. ■

Using the same ideas as in this proof, the reader can prove the following range characterization.

Theorem 5.4. *The timelike Radon transform maps bijectively the closure of the span of functions*

$$f_{\ell,m}(\varphi, r) = \cosh r \tanh^{2\ell} r \sin(m\varphi),$$

where $\ell \leq [(m-1)/2]$ *into the set of functions of the form*

$$F(\varphi, p) = \sqrt{1-p^2} \sum_{m=0}^{\infty} \varepsilon_m(m\varphi) \sum_{i=0}^{[(m-1)/2]} d_{m,m-1-2i} p^{m-1-2i},$$

where $d_{m,m-1-2i}$ *are arbitrary constants and*

$$\varepsilon_m(\psi) = \begin{cases} \cos(\psi) & \text{if } m \text{ is even} \\ \sin(\psi) & \text{if } m \text{ is odd.} \end{cases}$$

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