Local geometric loops

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Abstract. In this paper a detailed analysis of the geometric local loops is given. Among others, we show a geometric local loop, that is not a group, having a local one parameter group in every direction.

1. Definition

Let $\mathcal{U}$ and $\mathcal{V}$ be two n-dimensional C^1 hypersurfaces in $\mathbb{R}^{n+1}$ through O, the origin, and through $e = (0, 0, \ldots, 0, 1)$, the "unit". We define a local loop multiplication "$\circ$" on $\mathbb{R}^n$ embedded, as a subspace, in $\mathbb{R}^{n+1}$ orthogonally to e through the origin as a subspace.

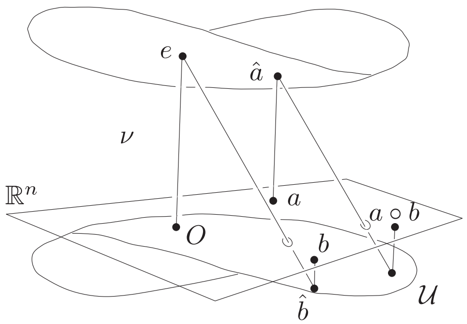

Figure 1.

Let a and b be two points on the embedded $\mathbb{R}^n$ and let $\hat{a}$ and $\hat{b}$ be the orthogonal projections of a and b onto $\mathcal{V}$ and $\mathcal{U}$. Let L be the straight line through $\hat{a}$ with the same direction as the segment $e\hat{b}$. Now the projection of the intersection of L and $\mathcal{U}$ into the embedded $\mathbb{R}^n$ will give $a \circ b$. If the hypersurfaces $\mathcal{U}$ and $\mathcal{V}$ are

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locally given by the functions $u, v\colon \mathbb{R}^n \to \mathbb{R}$, using these functions one can easily reformulate this definition as

$$(u(b)-1)(a\circ b-a)=(u(a\circ b)-v(a))b, \tag{1}$$

where a, b and $a\circ b$ are vectors of $\mathbb{R}^n$ and $v(a)$, $u(b)$ and $u(a\circ b)$ are scalars.

Obviously from the definition, zero is the unit element of this operation. Also, as the construction easily shows, it has inverse on both sides. Hence this is really a loop. Similar definition can be found in [1], where a triple of hypersurfaces in the projective space is used.

2. One dimension

Here we determine all the one-parameter local geometric loops, that are groups. This is important, because any two-dimensional plane containing the unit vector e and the origin cuts out a one parameter local geometric loop from the higher dimensional local geometric loops. This shows also that any local geometric loop has one-parameter local geometric subloop in each direction.

Theorem 2.1. *A local geometric loop in a neighborhood of the zero in $\mathbb{R}$ is a local group if and only if*

$$u(x)=\frac{1}{2}+ax-\frac{1}{2}\sqrt{1+bx+cx^2}\quad \text{and}\quad v(x)=\frac{1}{2}+ax+\frac{1}{2}\sqrt{1+bx+cx^2}$$

for some constants a, b and c. The corresponding operations are the followings: If $c=b^2/4$ then

$$x\circ y=x+y+\frac{b}{2}xy;$$

if $c<0$ then

$$x\circ y=\frac{b}{2c}-\frac{\sqrt{-c-b^2/4}}{c}\times$$
$$\times\sinh\left(\operatorname{arcsinh}\frac{b/2-cx}{\sqrt{-c-b^2/4}}+\operatorname{arcsinh}\frac{b/2-cy}{\sqrt{-c-b^2/4}}-\operatorname{arcsinh}\frac{b/2}{\sqrt{-c-b^2/4}}\right),$$

if $c>0$ then

$$x\circ y=\frac{b}{2c}+\frac{\sqrt{c+b^2/4}}{c}\times$$
$$\times\sin\left(\arcsin\frac{cx-b/2}{\sqrt{c+b^2/4}}+\arcsin\frac{cy-b/2}{\sqrt{c+b^2/4}}+\arcsin\frac{b/2}{\sqrt{c+b^2/4}}\right),$$

if $c=0$ and $b\neq 0$ then

$$x\circ y=x+y+\frac{2}{b}\left(1-\sqrt{1+bx}\right)\left(1-\sqrt{1+by}\right).$$

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Proof. Let $f(x,y) = x \circ y$ be the loop multiplication which is defined by the geometric construction:

$$(f(x,y)-x)(u(y)-1) = y(u(f(x,y))-v(x)). \tag{1'}$$

Differentiating this with respect to y at zero we get

$$-\partial_2 f(x,0) = u(x) - v(x). \tag{2'}$$

Differentiating twice with respect to y at zero we obtain $-\partial_2^2 f(x,0) = 2(\dot{u}(x) - \dot{u}(0))\partial_2 f(x,0)$, hence we conclude that

$$u(a) = a\dot{u}(0) - \int_0^a \frac{\partial_2^2 f(x,0)}{2\partial_2 f(x,0)}dx. \tag{2}$$

If f is a group multiplication then from the associativity it follows that $f(x,f(y,z)) = f(f(x,y),z)$. Differentiating first by y at zero and then by z at zero we obtain

$$\partial_2^2 f(x,0) + \partial_2 f(x,0)\partial_1\partial_2 f(0,0) = \partial_1\partial_2 f(x,0)\partial_2 f(x,0).$$

Using in (2) the expression of $\partial_2^2 f(x,0)/\partial_2 f(x,0)$ given by this equation and carrying out the integration yield

$$u(a) = \frac{1}{2} + a\Big(\dot{u}(0) + \frac{1}{2}\partial_1\partial_2 f(0,0)\Big) - \frac{1}{2}\partial_2 f(a,0). \tag{3}$$

Thus to prove the theorem we only have to calculate $\partial_2 f(x,0)$. To this end differentiate the identity of associativity with respect to x, y and z at zero. Three equations are obtained:

$$\begin{aligned} \partial_1 f(0,f(y,z)) &= \partial_1 f(y,z)\partial_1 f(0,y) \\ \partial_2 f(x,y)\partial_2 f(y,0) &= \partial_2 f(f(x,y),0) \\ \partial_2 f(x,z)\partial_1 f(0,z) &= \partial_1 f(x,z)\partial_2 f(x,0). \end{aligned} \tag{4}$$

Using the notation $\psi(x) = \partial_1 f(0,x)/\partial_2 f(x,0)$ these equations imply $\psi(f(x,y)) = \psi(x)\psi(y)$. Differentiating this equation by x at zero and by y at zero and taking the difference of the two equations just obtained, one gets

$$\dot{\psi}(x)(\partial_1 f(0,x) - \partial_2 f(x,0)) = 0.$$

This means $\frac{d}{dx}\left((\psi(x)-1)^2\right) = 0$. Since $\psi(0) = 1$ we are lead to $\psi \equiv 1$, i.e. $\partial_1 f(0,x) \equiv \partial_2 f(x,0)$.

Let F be the function for which $\dot F(x) = 1/\partial_1 f(0,x)$ and let

$$\phi(x,y) = F(f(x,y)) - F(x) - F(y) + F(0).$$

Then $\phi(0,0) = 0$. From the equation (4) and the equation $\partial_1 f(0,x) = \partial_2 f(x,0)$ one can easily verify $\partial_1\phi \equiv \partial_2\phi \equiv 0$. Thus $\phi \equiv 0$ and if we let $\varphi(x) = F(x) - F(0)$ we obtain $f(x,y) = \bar\varphi(\varphi(x)+\varphi(y))$, where $\bar\varphi$ is the inverse of φ. The inverse $\bar\varphi$ exists, of course, in a neighborhood of the zero, because $\dot\varphi(0) = 1$. (Note that this yields $x \circ y = y \circ x$ that is a one-parameter C^1 local geometric group is always commutative!)

Using this expression of f in (3) and then applying the result and (2') in (1') we get

$$y\Big(\dot{\bar\varphi}(\varphi(x)+\varphi(y)) + \frac{1}{\dot\varphi(x)}\Big) = (\bar\varphi(\varphi(x)+\varphi(y)) - x)\Big(1 + \frac{1}{\dot\varphi(y)}\Big). \tag{5}$$

Taking the difference of this equation and the one obtained by exchanging of x and y results in

$$(\dot{\bar\varphi}(\varphi(x)+\varphi(y)) - 1)\,(x-y) = \bar\varphi(\varphi(x)+\varphi(y))\Big(\frac{1}{\dot\varphi(x)} - \frac{1}{\dot\varphi(y)}\Big).$$

Differentiation with respect to x at zero leads to

$$y\ddot{\bar\varphi}(\varphi(y)) + 1 = \frac{\dot{\bar\varphi}(\varphi(y))}{\dot\varphi(y)} + y\ddot\varphi(0).$$

Taking into account that

$$\ddot{\bar\varphi}(\varphi(y))\dot\varphi(y) = \frac{d}{dy}(\dot{\bar\varphi}(\varphi(y))) = \frac{d}{dy}\Big(\frac{1}{\dot\varphi(y)}\Big) = \frac{-\ddot\varphi(y)}{(\dot\varphi(y))^2}$$

and simplifying the expression, our equation becomes

$$\frac{2(y\ddot\varphi(0) - 1)}{y} = -2\frac{\frac{d}{dy}(y\dot\varphi(y))}{(y\dot\varphi(y))^3}.$$

An easy integration results in

$$\frac{1}{\dot\varphi(x)} = \sqrt{1 - 2\ddot\varphi(0)x + cx^2}$$

for some constant c. To complete the proof, since $\partial_2 f(x,0) = 1/\dot\varphi(x)$, we only have to calculate φ and then verify (5). Since this task is straightforward and rather tedious we only sketch the way how to do it and only in the case when $c < 0$.

Integrating $\dot{\varphi}$ using the notation $b = -2\ddot{\varphi}(0)$ one gets

$$\varphi(x) = \frac{1}{\sqrt{-c}}\Big(\operatorname{arcsinh}\frac{b/2 - cx}{\sqrt{-c - b^2/4}} - \operatorname{arcsinh}\frac{b/2}{\sqrt{-c - b^2/4}}\Big)$$

and then

$$\bar{\varphi}(x) = \frac{b}{2c} - \frac{\sqrt{-c - b^2/4}}{c}\sinh\Big(x\sqrt{-c} + \operatorname{arcsinh}\frac{b/2}{\sqrt{-c - b^2/4}}\Big).$$

Applying these in (5) and using the notations $\hat{x} = \frac{(b/2 - cx)}{\sqrt{-c-b^2/4}}$, $\hat{y} = \frac{(b/2 - cy)}{\sqrt{-c-b^2/4}}$ and $\hat{o} = \frac{(b/2)}{\sqrt{-c-b^2/4}}$ we obtain to verify the equation

$$\begin{aligned}(\hat{y} - \hat{o})(\cosh(\operatorname{arcsinh}\hat{x} + \operatorname{arcsinh}\hat{y} - \operatorname{arcsinh}\hat{o}) + \cosh(\operatorname{arcsinh}\hat{x}))\\ = (\cosh(\operatorname{arcsinh}\hat{y}) + \cosh(\operatorname{arcsinh}\hat{o}))\times\\ \times(\sinh(\operatorname{arcsinh}\hat{x} + \operatorname{arcsinh}\hat{y} - \operatorname{arcsinh}\hat{o}) - \hat{x}).\end{aligned}$$

This can be justified by using the addition theorems for the hyperbolic functions after substituting $\hat{y} = \sinh(\operatorname{arcsinh}\hat{y})$, $\hat{o} = \sinh(\operatorname{arcsinh}\hat{o})$ and $\hat{x} = \sinh(\operatorname{arcsinh}\hat{x})$. ■

3. Higher dimensions

In this section we investigate two algebraic properties of a local geometric loop. We restrict ourself to higher dimensions that allows the use of simple geometric ideas. As we shall see there is a significiant difference between the higher dimensional and the one dimensional local geometric loops. The algebraic properties are much stronger in higher dimension.

Theorem 3.1. *A local geometric loop in a neighborhood of the zero in* $\mathbb{R}^n$ $(n \geq 2)$ *is local group if and only if* $\mathcal{U}$ *and* $\mathcal{V}$ *are hyperplanes. The operation then is* $x \circ y = x + y + y\langle x, \dot{v}(0) - \dot{u}(0)\rangle$*, where* $\langle .,. \rangle$ *denotes the standard inner product.*

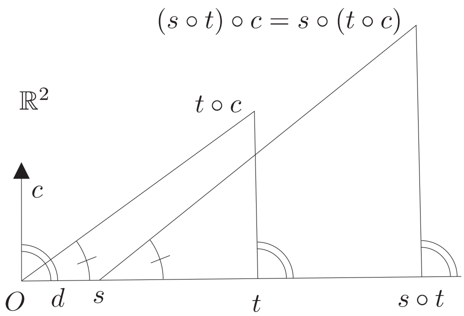

Figure 2.

Proof. It is enough to prove for dimension $n = 2$ because a geometric loop always has k-dimensional subloop in every direction for any $1 \leq k \leq n$. Furthermore, if the loop is group, then its subloops must be groups too. Thus from now on $n = 2$, the operation "$\circ$" is associative and sometimes denoted as a function $f(x, y) = x \circ y$.

Let d be a unit vector $s = \sigma d$, $t = \tau d$ $(\sigma, \tau \in \mathbb{R})$ and c an arbitrary vector not parallel to d in $\mathbb{R}^2$.

By the geometric construction of the loop, the identically noted angels on Figure 2 are equal. Hence the triangle $\{0, t, t \circ c\}$ is similar to the triangle $\{s, s \circ t, (s \circ t) \circ c\}$ hence the associativity implies

$$s \circ (t \circ c) = s + t \circ c \frac{|s \circ t - s|}{|t|}. \tag{6}$$

From $t \to 0$ it follows that $s \circ c = s + c|D_d f(s, 0)|$, where D_d denotes the differentiation of the second variable in direction d. But the one-dimensional local geometric groups are commutative (as we have shown in the previous proof) so if $s = \sigma d$ and $c = \gamma d$, in which case our equation remains valid by the continuity, we get $\sigma + \gamma|D_d f(s, 0)| = \gamma + \sigma|D_d f(c, 0)|$. This implies that

$$|D_d f(s, 0)| - 1 = \sigma \lim_{\gamma \to 0} \frac{|D_d f(c, 0)| - 1}{\gamma} = \sigma k(d)$$

for some real function k on the unit circle in $\mathbb{R}^2$. Thus we obtain $s \circ c = s + c + c|s|k(d)$. This is the first case in our previous theorem. Therefore $u(\gamma d) = \gamma g(d)$ and $v(\gamma d) = 1 + \gamma h(d)$ for some functions g and h on the unit circle. Since u and v are differentiable at the origin, $g(d) = \langle d, \dot{u}(0) \rangle$ and $h(d) = \langle d, \dot{v}(0) \rangle$, which proves the necessity of our condition.

To prove the sufficiency we write $u(s) = \langle s, U \rangle$ and $v(s) = 1 + \langle s, V \rangle$ in (1). This gives

$$(s \circ c - s)(\langle c, U \rangle - 1) = c(\langle s \circ c, U \rangle - 1 - \langle s, V \rangle),$$

hence $s \circ c = s + c\ell(s,c)$ for some function $\ell: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$. Substituting this expression into the above equation, we obtain $\ell(s,c) = 1 + \langle s, V - U\rangle$ as we stated. The verification of the associativity is left to the reader.

■

Theorem 3.2. *A local geometric loop in a neighborhood of the zero in* $\mathbb{R}^n$ $(n \geq 2)$ *has left inverse property* (*i.e.* $x \circ (\bar{x} \circ y) = (x \circ \bar{x}) \circ y = y$, *where* $x \circ \bar{x} = 0$) *if and only if* $\mathcal{U}$ *is hyperplane.*

Proof. The left inverse property in the case of $\bar{t} = s$ is given by (6) that gives $c - \bar{t} = t \circ c|\bar{t}|/|t|$. Using this equation and observing $s|\bar{s}| + \bar{s}|s| = 0$ we obtain

$$s \circ c = (c - \bar{s})\frac{|s|}{|\bar{s}|} = s + c\frac{|s|}{|\bar{s}|}.$$

As one can easily verify by a simple calculation this equation is not only necessary but also sufficient for the left inverse property. Substituting it into (1) implies

$$\frac{|s|}{|\bar{s}|}(u(c) - 1) = u\Big(s + c\frac{|s|}{|\bar{s}|}\Big) - v(s).$$

Now letting $c = \gamma d$, where $\gamma \in \mathbb{R}$ and $d \in \mathbb{R}^n$ and differentiating by γ at the zero we get $\langle \dot{u}(0), d\rangle = \langle \dot{u}(s), d\rangle$. Since the vector d is arbitrary, this means $\dot{u}(s)$ is a constant vector, which proves that $\mathcal{U}$ is a hyperplane.

To show the converse statement let $\mathcal{U}$ be a hyperplane. Then we have a vector U for which $u(x) = \langle U, x\rangle$. By (1) this gives

$$(x \circ y - x)(\langle U, y\rangle - 1) = y(\langle U, x \circ y\rangle - v(x)).$$

Letting $y = \bar{x}$ this gives $v(x) = \langle U, x\rangle + |x|/|\bar{x}|$. After some simplifications one gets

$$(\langle U, y\rangle - 1)\Big(x \circ y - x - y\frac{|x|}{|\bar{x}|}\Big) = y\Big\langle U, x \circ y - x - y\frac{|x|}{|\bar{x}|}\Big\rangle.$$

Let $\ell = x \circ y - x - y\frac{|x|}{|\bar{x}|}$. Taking the inner product of both sides with U we see $\langle U, \ell\rangle = 0$, which shows in the equation above that $\ell = 0$. Thus, $x \circ y = x + y\frac{|x|}{|\bar{x}|}$ which justifies the left inverse property.

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We note that the right inverse property seems to be much more complicated and we do not have any definitive result. The following is easy to prove using our above results.

Corollary 3.3. *If a local geometric loop has the left inverse property and all its one-parameter subloops are groups then it is a group.*

The following corollary shows the force of the left inverse property indirectly. In other words, the left inverse property in Corollary 3.3 is necessary.

Corollary 3.4. *There are geometric local loops in a neighborhood of the zero in* $\mathbb{R}^n$ $(n \geq 2)$*, that are not groups, but all their one dimensional subloops, that exist in every directions, are groups.*

Because the proof is obvious, we only display the defining functions

$$u(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1 + c|x|^2} \quad \text{and} \quad v(x) = \frac{1}{2} + \frac{1}{2}\sqrt{1 + c|x|^2},$$

where $x \in \mathbb{R}^n$ is in a sufficiently small neighborhood of the zero, and c is a small real number.

This corollary is somewhat surprising, because Kuz'min [2] proved that the power associative loops have one parameter subgroups in every direction. One could think that the geometric property is so strong that in this case Kuz'min's theorem can be reversed. But this is not the case.

References

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