A CONVEX COMBINATORIAL PROPERTY OF COMPACT SETS IN THE PLANE AND ITS ROOTS IN LATTICE THEORY

GÁBOR CZÉDLI AND ÁRPÁD KURUSA

Dedicated to George A. Grätzer
on the occasion of the fifty-fifth anniversary of the Grätzer–Schmidt Theorem and the fortieth anniversary of his monograph "General Lattice Theory"

Abstract. K. Adaricheva and M. Bolat have recently proved that if $\mathcal{U}_0$ and $\mathcal{U}_1$ are circles in a triangle with vertices A_0, A_1, A_2, then there exist $j \in \{0,1,2\}$ and $k \in \{0,1\}$ such that $\mathcal{U}_{1-k}$ is included in the convex hull of $\mathcal{U}_k \cup (\{A_0, A_1, A_2\} \setminus \{A_j\})$. One could say disks instead of circles. Here we prove the existence of such a j and k for the more general case where $\mathcal{U}_0$ and $\mathcal{U}_1$ are compact sets in the plane such that $\mathcal{U}_1$ is obtained from $\mathcal{U}_0$ by a positive homothety or by a translation. Also, we give a short survey to show how lattice theoretical antecedents, including a series of papers on planar semimodular lattices by G. Grätzer and E. Knapp, lead to our result.

1. Aim and outline

Our goal. Apart from a survey of historical nature, which the reader can skip over if he is interested only in the main result, this paper belongs to elementary combinatorial *geometry*. The motivation and the excuse that this paper is submitted to CGASA are the following. The first author has recently written a short biographical paper [16] to celebrate professor George A. Grätzer, and also an interview [17] with him; at the time of this writing, both have already appeared online in CGASA. Although [16] mentions that G. Grätzer's purely lattice theoretical results have lead to results in geometry, no detail on the transition from lattice theory to geometry is given there. This paper, besides presenting a recent result in geometry, exemplifies how such a purely lattice theoretical target as studying congruence lattices of finite lattices can lead, surprisingly, to some progress in geometry.

The real plane and the usual convex hull operator on it will be denoted by $\mathbb{R}^2$ and $\mathrm{Conv}_{\mathbb{R}^2}$. In order to formulate our result, we need the following two kinds of planar transformations, that is, $\mathbb{R}^2 \to \mathbb{R}^2$ maps. Given $P \in \mathbb{R}^2$ and $0 < \lambda \in \mathbb{R}$, the *positive homothety* with (homothetic) center P and ratio λ is defined by

$$\chi_{P,\lambda} \colon \mathbb{R}^2 \to \mathbb{R}^2 \text{ by } X \mapsto (1-\lambda)P + \lambda X = P + \lambda(X - P). \quad (1.1)$$ ⟨12, 17⟩

Date: CET 03:20, July 10, 2018.
2010 *Mathematics Subject Classification.* Primary 52C99, secondary 52A01, 06C10.
Key words and phrases. Congruence lattice, planar semimodular lattice, convex hull, compact set, circle, combinatorial geometry, abstract convex geometry, anti-exchange property.

The more general concept of homotheties where λ can also be negative is not needed in the present paper. For a given $P \in \mathbb{R}^2$, the map $\mathbb{R}^2 \to \mathbb{R}^2$, defined by $X \mapsto P + X$, is a *translation*. Our main goal is to prove the following theorem.

Theorem 1.1. *Let $A_0, A_1, A_2 \in \mathbb{R}^2$ be points of the plane $\mathbb{R}^2$. Also, let $\mathcal{U}_0 \subset \mathbb{R}^2$ and $\mathcal{U}_1 \subset \mathbb{R}^2$ be compact sets such that at least one of the following three conditions holds:*

(a) *$\mathcal{U}_1$ is a positive homothetic image of $\mathcal{U}_0$, that is, $\mathcal{U}_1 = \boldsymbol{\chi}_{P,\lambda}(\mathcal{U}_0)$ for some $P \in \mathbb{R}^2$ and $0 < \lambda \in \mathbb{R}$;*
(b) *$\mathcal{U}_1$ is obtained from $\mathcal{U}_0$ by a translation;*
(c) *at least one of $\mathcal{U}_0$ and $\mathcal{U}_1$ is a singleton.*

With these assumptions,

$$\begin{cases} \textit{if } \mathcal{U}_0 \cup \mathcal{U}_1 \subseteq \mathrm{Conv}_{\mathbb{R}^2}(\{A_0, A_1, A_2\}), \textit{ then there exist subscripts} \\ j \in \{0,1,2\} \textit{ and } k \in \{0,1\} \textit{ such that} \\ \qquad \mathcal{U}_{1-k} \subseteq \mathrm{Conv}_{\mathbb{R}^2}\big(\mathcal{U}_k \cup (\{A_0, A_1, A_2\} \setminus \{A_j\})\big). \end{cases} \tag{1.2}$$

⟨2, 8, 9, 11

Since at least one of the conditions 1.1(a), 1.1(b), and 1.1(c) holds for any two circles, the following result of Adaricheva and Bolat becomes an immediate consequence of Theorem 1.1.

Corollary 1.2 (Adaricheva and Bolat [2, Theorem 3.1])**.** *Let $\mathcal{U}_0$ and $\mathcal{U}_1$ be* circles *in the plane $\mathbb{R}^2$. Then (1.2) holds for all $A_0, A_1, A_2 \in \mathbb{R}^2$.*

Four comments are appropriate here. First, according to another terminology, positive homotheties (in our sense) and translations generate the group of positive *homothety-translations*, and one might think of using homothety-translations in Theorem 1.1. It will be pointed out in Lemma 3.9 that we would not obtain a new result in this way, because a homothety-translation is always a homothety or a translation. Since the rest of the paper focuses mainly on homotheties in our sense, we use disjunction rather than the hyphened form "homothety-translation". Second, it is easy to see that (1.2) does not hold for two *arbitrary* compact sets, so the disjunction of (a), (b), and (c) cannot be omitted from Theorem 1.1; see also Czédli [14] for related information. Third, Example 4.1 of Czédli [15] rules out the possibility of generalizing Theorem 1.1 for higher dimensions. Fourth, one may ask whether stipulating the *compactness* of $\mathcal{U}_0$ and $\mathcal{U}_1$ is essential in Theorem 1.1. Clearly, "compact" could be replaced by "(topologically) closed", since a non-compact closed set cannot be a subset of the triangle $\mathrm{Conv}_{\mathbb{R}^2}(\{A_0, A_1, A_2\})$, but this trivial rewording would not be a valuable improvement. The situation

$$\begin{aligned} &A_0 := \langle 6, 0\rangle,\ \ A_1 := \langle -3, 3\sqrt{3}\rangle,\ \ A_2 := \langle -3, -3\sqrt{3}\rangle, \\ &\mathcal{U}_0 := \{\langle x,y\rangle : x^2+y^2 < 1\} \cup \{\langle x,y\rangle : x^2+y^2 = 1 \text{ and } x \text{ is rational}\}, \\ &\mathcal{U}_1 := \{\langle x,y\rangle : x^2+y^2 < 1\} \cup \{\langle x,y\rangle : x^2+y^2 = 1 \text{ and } x \text{ is irrational}\}, \end{aligned}$$

exemplifies that Theorem 1.1 would fail without requiring the compactness of $\mathcal{U}_0$ and $\mathcal{U}_1$.

Prerequisites and outline. No special prerequisites are required; practically, every mathematician with usual M.Sc. background can understand the proof of Theorem 1.1. On the other hand, Section 2 is of historical nature and can be interesting mainly for specialists.

The rest of the paper is structured as follows. In Section 2, starting from lattice theoretical results including Grätzer and Knapp [41, 42, 43, 44, 45], we survey how lattice theoretical results lead to the present paper. Also, we say a few words on some similar results that belong to combinatorial geometry. Section 3 proves Theorem 1.1 only for the particular case where the convex closures of the compact sets $\mathcal{U}_0$ and $\mathcal{U}_1$ are "edge-free" (to be defined later). Section 4 reduces the general case to the edge-free case and completes the proof of Theorem 1.1.

2. From George Grätzer's lattice theoretical papers to geometry

Congruence lattices of finite lattices are well known to form George Grätzer's favorite research topic, in which he has proved many nice and deep results; see, for example, the last section in the biographical paper [16] by the first author. At first sight, it is not so easy to imagine interesting links between this topic and geometry. The aim of this section is to present such a link by explaining how some of Grätzer's *purely lattice theoretical* results have lead to the present paper and other papers in *geometry*. Instead of over-packing this section with too many definitions and statements, we are going to focus on *links* connecting results and publications. This explains that, in this section, some definitions are given only after discussing the links related to them. Although a part of this exposition is based on the experience of the first author, the link between lattice theory and the geometrical topic of the present paper is hopefully more than just a personal feeling.

2.1. **Planar semimodular lattices and their congruences.** On November 28, 2006, Grätzer and his student, Edward Knapp submitted their first paper, [41], to Acta Sci. Math. (Szeged) on planar semimodular lattices. A lattice $\mathcal{L} = \langle L; \vee, \wedge\rangle$ is *semimodular* if, for every $a \in L$, the map $L \to L$, defined by $x \mapsto a \vee x$, preserves the "covers or equal" relation $\preceq$; as usual, $a \preceq b$ stands for $|\{x \in L : a \leq x \leq b\}| \in \{1, 2\}$. A lattice is *planar* if it is finite and has a Hasse diagram that is also a planar graph. Their first paper, [41], were soon followed by Grätzer and Knapp [42, 43, 44, 45]. After giving a structural description of planar semimodular lattices, they proved nice results on the congruence lattices of these lattices in [42], [44], and [45].

The lattice $\mathrm{Con}(\mathcal{L})$ of all congruence relations of $\mathcal{L}$ is the *congruence lattice* of $\mathcal{L}$, and it is known to be a distributive algebraic lattice by an old result of Funayama and Nakayama [35]. It is a milestone in the history of lattice theory that not every distributive lattice $\mathcal{D}$ can be represented in the form of $\mathrm{Con}(\mathcal{L})$; this famous result is due to Wehrung [59]. However,

$$\begin{cases} \text{every finite distributive lattice } \mathcal{D} \text{ can be represented, up} \\ \text{to isomorphism, as } \mathrm{Con}(\mathcal{L}) \text{ where } \mathcal{L} \text{ is a finite lattice.} \end{cases} \tag{2.1}$$ ⟨4⟩

This result is due to Dilworth, see [6], but it was not published until Grätzer and Schmidt [47]. There are several ways of generalizing (2.1); the first four of the following targets are due to G. Grätzer or to G. Grätzer and E. T. Schmidt.

- (T1) Find an $\mathcal{L}$ with nice properties in addition to $\mathrm{Con}(\mathcal{L}) \cong \mathcal{D}$,
- (T2) find an $\mathcal{L}$ of size being as small as possible,
- (T3) represent two or even more finite distributive lattices and certain isotone maps among them simultaneously,
- (T4) represent a finite ordered set (also known as a poset) as the ordered set of *principal congruences* of a finite lattice, and
- (T5) combine some of the targets above.

There are dozens of results and papers addressing these targets. The monograph Grätzer [36] surveyed the results of this kind available before 2006. Ten years later, the new edition [40] became much more extensive, and the progress has not yet finished. The series of papers by Grätzer and Knapp fits well into the targets listed above. Indeed, [44] fits (T1) by providing a *rectangular lattice* $\mathcal{L}$ while, fitting both (T1) and (T2), [45] minimizes the size of this rectangular $\mathcal{L}$. A *rectangular lattice* is a planar semimodular lattice with a pair $\langle u, v\rangle \neq \langle 0, 1\rangle$ of double irreducible elements such that $u \wedge v = 0$ and $u \vee v = 1$; these lattices have nice rectangle-shaped planar diagrams.

Next, in their 2010 paper, Grätzer and Nation [46] proved a stronger form of the classical Jordan–Hölder theorem for groups from the nineteenth century. Here we formulate their result only for groups, but note that both [46] and [25], to be mentioned soon, formulated the results for semimodular lattices. For subnormal subgroups $A \lhd B$ and $C \lhd D$ of a given group G, the quotient B/A is said to be *subnormally down-and-up projective* to D/C if there are subnormal subgroups $E \lhd F$ such that $AF = B$, $A \cap F = E$, $CF = D$, and $C \cap F = E$. Grätzer and Nation's result for finite groups says that whenever $\{1\} = X_0 \lhd X_1 \lhd \cdots \lhd X_n = G$ and $\{1\} = Y_0 \lhd Y_1 \lhd \cdots \lhd Y_m = G$ are composition series of a group G, then $n = m$ and

$$\begin{cases} \text{there exists a permutation } \pi\colon \{1, \dots, n\} \to \{1, \dots, n\} \text{ such that} \\ X_i/X_{i-1} \text{ is subnormally down-and-up projective to } Y_{\pi(i)}/Y_{\pi(i)-1} \\ \text{for all } i \in \{1, \dots, n\}. \end{cases} \tag{2.2}$$ ⟨4, 7⟩

(The original Jordan–Hölder theorem states only that the quotient groups X_i/X_{i-1} and $Y_{\pi(i)}/Y_{\pi(i)-1}$ are isomorphic, because they are in the *transitive closure* of subnormal down-and-up projectivity.) Not much later, Czédli and Schmidt [25] added that π in (2.2) is uniquely determined. The proof in [25] is based on *slim* planar semimodular lattices; this concept was introduced in Gätzer and Knapp

[41]: a planar semimodular lattice is *slim* if $\mathcal{M}_3$, the five-element nondistributive modular lattice, cannot be embedded into it in a cover-preserving way.

Next, (T1)–(T5) and the applicability of slim semimodular lattices for groups motivated further results on the structure of slim semimodular lattices, including Czédli [8], Czédli and Grätzer [19], Czédli, Ozsvárt and Udvari [24], and Czédli and Schmidt [26] and [27]. Some results on the congruences and congruence lattices of these lattices, including Czédli [9], [12], [13], Czédli and Makay [23], Grätzer [38], [39], and Grätzer and Schmidt [48] and [49] have also been proved. Neither of these two lists is complete; see the book sections Czédli and Grätzer [20] and Grätzer [37] and the monograph Grätzer [40] for additional information and references.

2.2. **Convex geometries as combinatorial structures.** It was an anonymous referee of Czédli, Ozsvárt and Udvari [24] who pointed out that slim semimodular lattices can be viewed as convex geometries of convex dimension at most 2; see Proposition 2.1 and the paragraph following it in the present paper. As the first consequence of this remark, Adaricheva and Czédli [3] and Czédli [10] gave a lattice theoretical new proof of the "coordinatizability" of convex geometries by permutations; the original combinatorial result is due to Edelman and Jamison [31].

As we know from Monjardet [55], convex geometries are so important that they had been discovered or rediscovered in many equivalent forms even by 1985 not only as combinatorial structures but also as lattices. This explains that the terminology is far from being unique. Here we go after the terminology used in Czédli [10] even when much older results are cited. If the reader is interested in further information on convex geometries, he may turn to [10] for a limited survey or to Adaricheva and Nation [5] for a more extensive treatise. To keep the size limited, we do not mention antimatroids and meet-distributivity; see the survey part of Czédli [10] for references on them.

In order to give a *combinatorial definition*, the *power set* of a given *finite* set E will be denoted by $\mathrm{Pow}(E) := \{X : X \subseteq E\}$. If a map $\Phi\colon \mathrm{Pow}(E) \to \mathrm{Pow}(E)$ satisfies the rules $X \subseteq \Phi(X) \subseteq \Phi(Y) = \Phi(\Phi(Y))$ for all $X \subseteq Y \subseteq E$, then Φ is a *closure operator* over the set E. A pair $\langle E; \Phi\rangle$ is a *convex geometry* if E is a nonempty set, Φ is a closure operator over E, $\Phi(\emptyset) = \emptyset$, and, for all $p, q \in E$ and $X = \Phi(X) \in \mathrm{Pow}(E)$, the *anti-exchange property*

$$(\ p \neq q,\ p \notin X,\ q \notin X,\ p \in \Phi(X \cup \{q\})\ ) \Rightarrow q \notin \Phi(X \cup \{p\}) \tag{2.3}$$

holds. For example, if E is a finite set of points of $\mathbb{R}^2$, then we obtain a convex geometry $\langle E; \Phi\rangle$ by letting

$$\Phi\colon \mathrm{Pow}(E) \to \mathrm{Pow}(E) \text{ defined by } \Phi(X) := E \cap \mathrm{Conv}_{\mathbb{R}^2}(X). \tag{2.4}$$ ⟨7⟩

The *dual* of a lattice $\mathcal{K} = \langle K; \vee, \wedge\rangle$ is denoted by $\mathcal{K}^{\mathrm{dual}} := \langle K; \wedge, \vee\rangle$. For $x \in \mathcal{K}$, let $x^* := \bigvee\{y : x \prec y\}$. Let $\mathcal{M}_3$ denote the five-element modular nondistributive lattice. By a *join-distributive lattice* we mean a semimodular lattice of finite length that does not include $\mathcal{M}_3$ a sublattice. (This concept should not be confused with join-semidistributivity.) Equivalently, a semimodular lattice $\mathcal{K}$ of finite length is *join-distributive* if the interval $[x, x^*]$ is a distributive lattice for all $x \in \mathcal{K} \setminus \{1\}$; this is the definition that explains the current terminology. From the literature, Czédli [10, Proposition 2.1] collects eight equivalent definitions of join-distributivity; the oldest one of them is due to Dilworth [29].

Given a convex geometry $\langle E; \Phi\rangle$, the set $\{X \in \mathrm{Pow}(E) : X = \Phi(X)\}$ of *closed sets* forms a lattice with respect to set inclusion $\subseteq$. The *dual* of this lattice will be denoted by $\mathcal{L}_{\mathrm{lat}}(\langle E; \Phi\rangle)$. As usual, a set $X \in \mathrm{Pow}(E)$ is called *open* if $E \setminus X$ is closed. With this terminology, $\mathcal{L}_{\mathrm{lat}}(\langle E; \Phi\rangle)$ can be considered as the lattice of open subsets of E with respect to set inclusion $\subseteq$.

For a finite lattice $\mathcal{K}$, the set $J(\mathcal{K})$ of (non-zero) *join-irreducible elements* is defined as $\{x \in \mathcal{K} : \text{there is exactly one } y \in \mathcal{K} \text{ with } y \prec x\}$. Next, for a finite lattice $\mathcal{L}$, we define a closure operator

$$\Phi_{\mathcal{L}^{\mathrm{dual}}} \colon \mathrm{Pow}(J(\mathcal{L}^{\mathrm{dual}})) \to \mathrm{Pow}(J(\mathcal{L}^{\mathrm{dual}})) \ \text{ by}$$
$$\Phi_{\mathcal{L}^{\mathrm{dual}}}(X) := \Big\{y \in J(\mathcal{L}^{\mathrm{dual}}) : y \leq_{\mathcal{L}^{\mathrm{dual}}} \bigvee_{\mathcal{L}^{\mathrm{dual}}} X\Big\}, \text{ and we let}$$
$$\mathcal{G}_{\mathrm{conv}}(\mathcal{L}) := \langle J(\mathcal{L}^{\mathrm{dual}}); \Phi_{\mathcal{L}^{\mathrm{dual}}}\rangle.$$

Of course, the inequality above is equivalent to $y \geq \bigwedge X$ in $\mathcal{L}$ and $J(\mathcal{L}^{\mathrm{dual}})$ equals the set of meet-irreducible elements of $\mathcal{L}$. The following proposition, cited as the combination of Proposition 7.3 and Lemma 7.4 in [10], is due to Adaricheva, Gorbunov, and Tumanov [4] and Edelman [30].

Proposition 2.1. *Let $\langle E; \Phi\rangle$ and $\mathcal{L}$ be a convex geometry and a join-distributive lattice, respectively. Then $\mathcal{L}_{\mathrm{lat}}(\langle E; \Phi\rangle)$ is a join-distributive lattice, $\mathcal{G}_{\mathrm{conv}}(\mathcal{L})$ is a convex geometry, and, in addition, we have that $\mathcal{G}_{\mathrm{conv}}(\mathcal{L}_{\mathrm{lat}}(\langle E; \Phi\rangle)) \cong \langle E; \Phi\rangle$ and $\mathcal{L}_{\mathrm{lat}}(\mathcal{G}_{\mathrm{conv}}(\mathcal{L})) \cong \mathcal{L}$.*

This proposition allows us to say that convex geometries and join-distributive lattices capture basically the same concept. Based on Proposition 2.1 and the theory of planar semimodular lattices summarized in Czédli and Grätzer [20], we can say that a convex geometry $\langle E; \Phi\rangle$ is of *convex dimension* at most 2 if $\mathcal{L}_{\mathrm{lat}}(\langle E; \Phi\rangle)$ is a slim semimodular lattice.

2.3. **From lattices to convex geometries by means of trajectories.** In order to describe the first step from Grätzer and Knapp [41, 42, 43, 44, 45] and Grätzer and Nation [46] towards geometry, we need to define trajectories. If $a \prec b$ in a finite lattice $\mathcal{K}$, then $[a, b]$ is called a *prime interval* of $\mathcal{K}$. The *set of prime intervals* of $\mathcal{K}$ will be denoted by $\mathrm{PrIntv}(\mathcal{K})$. Two prime intervals, $[a_0, b_0], [a_1, b_1] \in \mathrm{PrIntv}(\mathcal{K})$, are *consecutive* if $a_i = a_{1-i} \wedge b_i$ and $b_{1-i} = a_{1-i} \vee b_i$ hold for some $i \in \{0, 1\}$. The reflexive-transitive closure of consecutiveness is an equivalence relation on $\mathrm{PrIntv}(\mathcal{K})$, and its classes are called the *trajectories* of $\mathcal{K}$.

Trajectories were introduced in Czédli and Schmidt [25], and they played the key role in proving the uniqueness of π in (2.2). Soon afterwards, trajectories were intensively used when dealing with congruence lattices of slim planar semimodular lattices, because for $x \prec y$ and $a \prec b$ is such a lattice, one can describe with the help of trajectories whether the least congruence $\mathrm{con}(a,b)$ collapsing $\langle a,b\rangle$ contains (in other words, collapses) $\langle x,y\rangle$. Later, similarly to trajectories, a beautiful description of the containment $\langle x,y\rangle \in \mathrm{con}(a,b)$ was described by Grätzer's Swing Lemma; see Grätzer [38], and see Czédli, Grätzer, and Lakser [21] and Czédli and Makay [23] for a generalization and for alternative approaches. Note that Lemma 2.36 in Freese, Ježek, and Nation [34, page 41], which is due to Jónsson and Nation [51] originally, offers an alternative way to describe whether $\langle x,y\rangle \in \mathrm{con}(a,b)$.

For distinct prime intervals $[a_0,b_0],[a_1,b_1] \in \mathrm{PrIntv}(\mathcal{K})$, we say that $[a_0,b_0]$ and $[a_1,b_1]$ are *comparable* if either $b_0 \le a_1$, or $b_1 \le a_0$. It was proved in Adaricheva and Czédli [3] that

$$\left\{\begin{array}{l}\text{a finite semimodular lattice } \mathcal{L} \text{ is join-distributive if and only}\\ \text{if no two distinct comparable prime intervals of } \mathcal{L} \text{ belong to}\\ \text{the same trajectory.}\end{array}\right. \tag{2.5}$$ ⟨7⟩

Combining (2.5) with Proposition 2.1, we obtain a new description of convex geometries.

2.4. **Representing convex geometries.** Using the usual convex hull operator $\mathrm{Conv}_{\mathbb{R}^n}$ together with auxiliary points in a tricky way, Kashiwabara, Nakamura, and Okamoto [52] gave a representation theorem for convex geometries in 2005. The example described in (2.4) is simpler, but it is not appropriate to represent *every* convex geometry because of a very simple reason: if $\langle E;\Phi\rangle$ is a convex geometry of the form described in (2.4), then $J(\mathcal{L}_{\mathrm{lat}}(\langle E;\Phi\rangle))$ is an antichain, which is not so for every convex geometry. Hence, Czédli [11] introduced the following construction. Let E be a finite set of circles in the plane $\mathbb{R}^2$, and define a convex geometry $\langle E;\Phi\rangle$ where $\Phi\colon \mathrm{Pow}(E) \to \mathrm{Pow}(E)$ is defined by

$$\Phi(X) := \Big\{C \in E : C \subseteq \mathrm{Conv}_{\mathbb{R}^2}\Big(\bigcup_{D\in X} D\Big)\Big\}. \tag{2.6}$$ ⟨7, 8⟩

It is easy to see that we obtain a convex geometry in this way. Note, however, that (2.6) does not yield a convex geometry in general if, say, E is a set of triangles rather than a set of circles. After translating the problem to lattice theory with the help of Proposition 2.1 and using the toolkit developed for slim semimodular lattices in the papers mentioned in Subsection 2.1, Czédli [11] proved that

$$\left\{\begin{array}{l}\text{every convex geometry of convex dimension at most 2 can rep-}\\ \text{resented by circles in the sense of (2.6).}\end{array}\right. \tag{2.7}$$

In fact, [11] proves a bit more. While [11] is mainly a lattice theoretical paper, it was soon followed by two results with proofs that are geometrical. First, Richter and Rogers [56] represented every convex geometry analogously to (2.6) but using polygons instead of circles. Second, Czédli and Kincses [22] replaced polygons with objects taken from an appropriate family of so-called "almost circles". However, it was not known at that time whether circles would do instead of "almost circles".

2.5. **Some results of geometrical nature.** The problem whether every convex geometry can be represented by circles in the sense of (2.6) was solved in negative by Adaricheva and Bolat [2]. The main step in their argument is the proof of [2, Theorem 3.1]; see Corollary 1.2 here. In fact, they proved that (1.2), with self-explanatory syntactical refinements, holds even for arbitrary three *circles* A_0, A_1, A_2 and two additional circles, $\mathcal{U}_0, \mathcal{U}_1$. This is such an obstacle that does not allow to represent every convex geometry by circles. Even more is true; later, Kincses [53] found an Erdős–Szekeres type obstruction for representing convex geometries by *ellipses*. Similarly to ellipses, he could exclude many other shapes. On the positive side, Kincses [53] proved that every convex geometry can be represented by *ellipsoids* in $\mathbb{R}^n$ for some $n \in \mathbb{N}^+ := \{1, 2, 3, \dots\}$ in the sense of (2.6) with $\mathrm{Conv}_{\mathbb{R}^n}$ instead of $\mathrm{Conv}_{\mathbb{R}^2}$. However, it is not known whether n-dimensional balls could do instead of ellipsoids.

An earlier attempt to generalize Adaricheva and Bolat [2, Theorem 3.1], see Corollary 1.2 here, did not use homotheties and resulted in a new characterization of disks. Namely, for a convex compact set $\mathcal{U}_0 \subseteq \mathbb{R}^2$, Czédli [14] proved that

$$\begin{cases} \mathcal{U}_0 \text{ is a disk if and only if for every isometric copy } \mathcal{U}_1 \text{ of } \mathcal{U}_0 \text{ and} \\ \text{for any points } A_0, A_1, A_2 \in \mathbb{R}^2,\ (1.2) \text{ holds.} \end{cases} \tag{2.8}$$ ⟨8⟩

The condition on $\mathcal{U}_1$ above means that there exists a distance-preserving geometric transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathcal{U}_1 = \varphi(\mathcal{U}_0)$.

There are quite many known characterizations of circles and disks; we mention only one of them below. We say that $\mathcal{U}_0$ and $\mathcal{U}_1$ are *Fejes-Tóth crossing* if none of the sets $\mathcal{U}_0 \setminus \mathcal{U}_1$ and $\mathcal{U}_1 \setminus \mathcal{U}_0$ is path-connected. It was proved in Fejes-Tóth [33] that

$$\begin{cases} \text{a convex compact set } \mathcal{U}_0 \subseteq \mathbb{R}^2 \text{ is a disk if and only if} \\ \text{there is no isometric copy } \mathcal{U}_1 \text{ of } \mathcal{U}_0 \text{ such that } \mathcal{U}_0 \text{ and} \\ \mathcal{U}_1 \text{ are Fejes-Tóth crossing.} \end{cases} \tag{2.9}$$ ⟨8⟩

Motivated by the proof of (2.8), a more restrictive concept of crossing was introduced in Czédli [18]; it is based on properties of common supporting lines but we will not define it here. Replacing Fejes-Tóth crossing with "[18]-crossing", (2.9) turns into a stronger statement.

Finally, to conclude our mini-survey from George Grätzer's congruence lattices to geometry via a sequence of closely connected consecutive results, we note that Paul Erdős and E. G. Straus [32] extended (2.9) to an analogous characterization of balls in higher dimensions, but the "[18]-crossing" seems to work only in the plane $\mathbb{R}^2$.

3. Proofs for the edge-free case

As usual in lattice theory, $\mathcal{U} \subset \mathcal{V}$ means the conjunction of $\mathcal{U} \subseteq \mathcal{V}$ and $\mathcal{U} \neq \mathcal{V}$. If $\mathcal{V}$ is a *compact* subset of $\mathbb{R}^2$, then we often write $\mathcal{V} \subset \mathbb{R}^2$ since $\mathcal{V} \neq \mathbb{R}^2$ holds automatically. For compact sets $\mathcal{U}, \mathcal{V} \subseteq \mathbb{R}^2$,

$$\mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U} \cup \mathcal{V}) = \mathrm{Conv}_{\mathbb{R}^2}\big(\mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U}) \cup \mathcal{V}\big).$$

Hence, the inclusion in the third line of (1.2) is equivalent to the inclusion

$$\mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U}_{1-k}) \subseteq \mathrm{Conv}_{\mathbb{R}^2}\big(\mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U}_k) \cup (\{A_0, A_1, A_2\} \setminus \{A_j\})\big).$$

Also, if $\mathcal{U}$ is compact, then so is $\mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U})$; see, for example, the first sentence of the introduction in Hüsseinov [50]. Thus, it suffices to prove our theorem only for *convex* compact subsets of $\mathbb{R}^2$. Therefore, in the rest of the paper,

$$\begin{cases} \text{we will always assume that } \mathcal{U},\ \mathcal{U}_0, \text{ and } \mathcal{U}_1 \\ \text{are compact and } convex, \end{cases} \tag{3.1}$$ ⟨9, 23⟩

even if this is not repeated all the time.

The advantage of assumption (3.1) lies in the fact that the properties of planar convex compact sets are well understood. For example, if $\mathcal{U} \subset \mathbb{R}^2$ is such a set, then the boundary $\partial\mathcal{U}$ of $\mathcal{U}$ is known to be a simple closed continuous rectifiable curve; see Latecki, Rosenfeld, and Silverman [54, Thm. 32] and Topogonov [58, page 15]. Since the reader need not be a geometer, we note that all what we need to know about planar convex sets are surveyed in a short section of the open access paper Czédli and Stachó [28]. Some facts about these sets, however, are summarized in the next subsection for the reader's convenience.

3.1. Supporting lines and a comparison with the case of circles.

Let

$$\mathcal{C}_{\mathrm{unit}} \text{ denote the } unit\ circle\ \{\langle x, y\rangle : x^2 + y^2 = 1\}; \tag{3.2}$$

its elements are called *directions*. In the rest of the paper, we often assume that the lines ℓ in our considerations are *directed lines*; their directions are denoted by $\mathrm{dir}(\ell) \in \mathcal{C}_{\mathrm{unit}}$ and by arrows in our figures. A directed line ℓ determines two *closed* halfplanes; their intersection is ℓ. A subset of $\mathbb{R}^2$ is *on the left* of ℓ if each of its points belongs to the left closed halfplane. Points in the left halfplane of ℓ but not on ℓ are *strictly on the left* of ℓ; points *strictly on the right* of ℓ are defined analogously. We always assume that

$$\begin{cases} \text{a supporting line of a set } \mathcal{U} \text{ is directed, and it is} \\ \text{directed so that } \mathcal{U} \text{ is on its left.} \end{cases} \tag{3.3}$$ ⟨10⟩

Since every compact convex set in the plane is well known to be the intersection of the left halfplanes of its supporting lines, we have that

$$\begin{cases} \text{if a point } P \in \mathbb{R}^2 \text{ does } not \text{ belong to a compact convex set} \\ \mathcal{U}, \text{ then } \mathcal{U} \text{ has a directed supporting line } \ell \text{ such that } P \text{ is} \\ \text{strictly on the right of } \ell. \end{cases} \tag{3.4}$$ ⟨22⟩

If ℓ is a supporting line of a compact convex set $\mathcal{U}$, then the points of $\mathcal{U} \cap \ell$ are called *support points*. If ℓ is the only directed supporting line through a support point $P \in \mathcal{U} \cap \ell$, then ℓ is a *tangent line* and P is a *tangent point*. Otherwise, we say that P is a *vertex* of $\mathcal{U}$. The properties of directed supporting lines are summarized in the open access papers Czédli [14] and Czédli and Stachó [28], or in the more advanced treatise Bonnesen and Fenchel [7]. In particular, by a *pointed supporting line* of $\mathcal{U}$ we mean a pair $\langle P, \ell \rangle$ such that ℓ is a directed supporting line of $\mathcal{U}$ with support point P. In general, $\mathcal{U}$ may have pointed supporting lines $\langle P_1, \ell \rangle$ and $\langle P_2, \ell \rangle$ with the same line component but distinct support points $P_1 \neq P_2$.

In Czédli [15], which is devoted only to circles, there is a relatively short proof of Adaricheva and Bolat [2, Theorem 3.1], cited as Corollary 1.2 here. Most ideas of [15] are used in the present paper, but these ideas need substantial changes in order to overcome the following three difficulties: as opposed to circles, a compact convex set need not have a center with nice geometric properties, its boundary need not have a tangent line at each of its points, and the boundary can include straight line segments of positive lengths. In this section, we disregard the latter difficulty by calling a compact convex set $\mathcal{U}$ *edge-free* if no line segment of positive length is a subset of $\partial\mathcal{U}$. Equivalently, a compact convex set $\mathcal{U} \subset \mathbb{R}^2$ is said to be *edge-free* if $\ell \cap \mathcal{U}$ is a singleton (still equivalently, if $\ell \cap \partial\mathcal{U}$ is a singleton) for every supporting line ℓ of $\mathcal{U}$. Note that every singleton subset of $\mathbb{R}^2$ is an edge-free compact convex set. Let us emphasize that an edge-free set is *nonempty* by definition. In order to shed even more light on the concept just introduced, we formulate and prove an easy lemma.

Lemma 3.1. *A nonempty compact convex set $\mathcal{U}$ is edge-free if and only if $\ell \cap \partial\mathcal{U}$ consists of at most two points for every line ℓ.*

Proof. We can assume that $\mathcal{U}$ is not a singleton since otherwise the statement is trivial.

First, assume that $\mathcal{U}$ is *not* edge-free, and pick a supporting line ℓ of $\mathcal{U}$ with two distinct points, $P_1, P_2 \in \ell \cap \partial U$. Let $P_3 = (P_1 + P_2)/2$; it belongs to $\mathcal{U}$ by convexity. Since P_3 lies on a supporting line, it is not in the interior of $\mathcal{U}$. Hence, $P_1, P_2, P_3 \in \ell \cap \partial\mathcal{U}$, which shows that $\ell \cap \partial\mathcal{U}$ consists of more than two points; this implies the "if" part of the lemma.

Second, assume that $\mathcal{U}$ is edge-free and ℓ is a directed line in the plane; we need to show that $\ell \cap \partial\mathcal{U}$ consists of at most two points. Suppose the contrary, and let P_1, P_2, and P_3 be three distinct points of ℓ, in this order, such that they all belong also to $\partial\mathcal{U}$. Pick a supporting line ℓ_2 of $\mathcal{U}$ through P_2. Since $\mathcal{U}$ is edge-free, $\ell_2 \cap \partial\mathcal{U}$ is a singleton, whereby none of P_1 and P_3 lies on ℓ_2. Therefore, since P_2 is between P_1 and P_3, we have that $P_1 \in \mathcal{U}$ and $P_3 \in \mathcal{U}$ are strictly on different sides of ℓ_2; contradicting (3.3). □

Our target in the present section is to prove the following lemma.

Lemma 3.2 (Main Lemma). *If the points $A_0, A_1, A_2 \in \mathbb{R}^2$ and the* convex *compact sets $\mathcal{U}_0, \mathcal{U}_1 \subset \mathbb{R}^2$ from Theorem 1.1 satisfy at least one of the conditions* (a), (b), *and* (c) *given in the theorem and, in addition,*

(d) *$\mathcal{U}_0$ and $\mathcal{U}_1$ are* edge-free,

then implication (1.2) *holds.*

The proof of this lemma needs some preparation and auxiliary lemmas. In the rest of this section, we always assume that $\mathcal{U}_0$ and $\mathcal{U}_1$ are *edge-free*.

3.2. **Comets.** In this paper, the Euclidean distance $((P_x - Q_x)^2 + (P_y - Q_y)^2)^{1/2}$ of $P, Q \in \mathbb{R}^2$ is denoted by $\mathrm{dist}(P, Q)$. For nonempty *compact* sets $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^2$, $\mathrm{dist}(\mathcal{U}, \mathcal{V}) = \inf\{\mathrm{dist}(P, Q) : P \in \mathcal{U},\ Q \in \mathcal{V}\} = \min\{\mathrm{dist}(P, Q) : P \in \mathcal{U},\ Q \in \mathcal{V}\}$. For an edge-free compact convex set $\mathcal{U}$ with more than one elements and a point $F \in \mathbb{R}^2 \setminus \mathcal{U}$, we define the *comet* $\mathrm{Comet}(F, \mathcal{U})$ with *focus* F and *nucleus* $\mathcal{U}$ so that

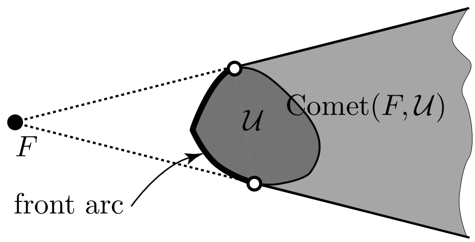

FIGURE 1. A comet

$$\mathrm{Comet}(F, \mathcal{U}) \text{ is the grey-filled area in Figure 1.} \tag{3.5}$$

More precisely, if we consider F as a source of light, then $\mathrm{Comet}(F, \mathcal{U})$ is the topological closure of the set of points that are shadowed by the nucleus $\mathcal{U}$. Note that $\mathcal{U}$, which is dark-grey in the figure, is a subset of $\mathrm{Comet}(F, \mathcal{U})$ and we have that $\mathrm{dist}(\{F\}, \mathrm{Comet}(F, \mathcal{U})) > 0$. As opposed to $\mathcal{U}$, $\mathrm{Comet}(F, \mathcal{U})$ is never compact.

Since $\mathcal{U}$ is compact, convex, and not a singleton, there are exactly two supporting lines of $\mathcal{U}$ through F, and they are supporting lines of $\mathrm{Comet}(F, \mathcal{U})$ as well. Since $\mathcal{U}$ is edge-free, these two lines are *tangent lines* of $\mathcal{U}$ and also of $\mathrm{Comet}(F, \mathcal{U})$. Each of these tangent lines has a unique tangent point on $\partial\mathcal{U}$. The arc of $\partial\mathcal{U}$ between these points that is closer to F is the *front arc* of the comet; see the thick curve in Figure 1. Note that the boundary of $\mathrm{Comet}(F, \mathcal{U})$ is the union of the front arc and two half-lines, so comets are never edge-free.

3.3. **Externally perspective compact convex sets.** For topologically closed convex sets $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathbb{R}^2$, we will say that

$$\mathcal{V}_1 \text{ is } \textit{loosely included} \text{ in } \mathcal{V}_2, \text{ in notation, } \mathcal{V}_1 \overset{\text{loose}}{\subset} \mathcal{V}_2, \tag{3.6}$$

if every point of $\mathcal{V}_1$ is an internal point of $\mathcal{V}_2$. The *interior* of a compact convex set $\mathcal{U}$ will be denoted by $\mathrm{Int}(\mathcal{U})$; note that $\mathrm{Int}(\mathcal{U}) = \mathcal{U} \setminus \partial\mathcal{U}$. Clearly, if $\mathcal{V}_1 \subset \mathbb{R}^2$ is compact, $\mathcal{V}_2 \subseteq \mathbb{R}^2$ is closed, and $P \in \mathrm{Int}(\mathcal{V}_1)$, then

$$\begin{cases} \mathcal{V}_1 \overset{\text{loose}}{\subset} \mathcal{V}_2 \text{ implies that there is a } \delta > 0 \text{ such that} \\ \chi_{P,1+\varepsilon}(\mathcal{V}_1) \overset{\text{loose}}{\subset} \mathcal{V}_2 \text{ for all positive } \varepsilon \le \delta, \end{cases} \tag{3.7}$$ ⟨20⟩

because $\mathbb{R}^2 \setminus \mathrm{Int}(\mathcal{V}_2)$ is closed and its distance from $\mathcal{V}_1$ is positive.

Next, for compact convex sets $\mathcal{U}_1, \mathcal{U}_2 \subset \mathbb{R}^2$, each of them with more than one element, we say that $\mathcal{U}_1$ and $\mathcal{U}_2$ are *externally perspective* if $\mathcal{U}_2 = \chi_{P,\lambda}(\mathcal{U}_1)$ for some (in fact, unique) $0 < \lambda \in \mathbb{R} \setminus \{1\}$ and $P \in \mathbb{R}^2 \setminus \mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U}_1 \cup \mathcal{U}_2)$; see (1.1). Equivalently, $\mathcal{U}_1$ and $\mathcal{U}_2$ are *externally perspective* if $\mathcal{U}_2 = \chi_{P,\lambda}(\mathcal{U}_1)$ with $P \notin \mathcal{U}_1$ and $0 < \lambda \ne 1$. Hence, by interchanging the subscripts if necessary, we will often assume that $0 < \lambda < 1$ if $\mathcal{U}_2 = \chi_{P,\lambda}(\mathcal{U}_1)$ is externally perspective to $\mathcal{U}_1$.

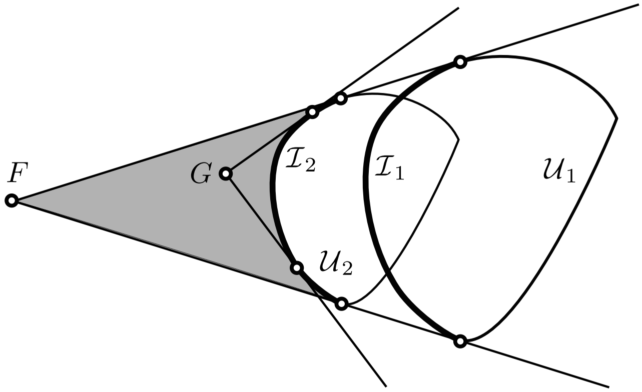

FIGURE 2. Illustration for Lemma 3.3

The following lemma is obvious by Figure 2.

Lemma 3.3. *Let $\mathcal{U}_1$ and $\mathcal{U}_2 = \chi_{F,\lambda}(\mathcal{U}_1)$ be externally perspective compact convex subsets of the plane such that $0 < \lambda < 1$. If G is an internal point of the grey-filled area surrounded by the common tangent lines of $\mathcal{U}_1$ and $\mathcal{U}_2$ through F and the front arc $\mathcal{I}_2$ of* $\mathrm{Comet}(F,\mathcal{U}_2)$, *then* $\mathrm{Comet}(F,\mathcal{U}_1)$ *is loosely included in* $\mathrm{Comet}(G,\mathcal{U}_2)$.

In the rest of the paper, to ease the notation,

$$\triangle_{A_0,A_1,A_2} \text{ will stand for } \mathrm{Conv}_{\mathbb{R}^2}(\{A_0, A_1, A_2\}). \tag{3.8}$$ ⟨17, 20⟩

Next, as a "loose counterpart" of the 2-Carousel Rule defined in Adaricheva [1], we formulate the following lemma.

Lemma 3.4. *Let A_0, A_1, and A_2 be non-collinear points in the plane. If B_0 and B_1 are* distinct internal *points of $\triangle_{A_0,A_1,A_2}$, then there exist $j \in \{0, 1, 2\}$ and*

$k \in \{0, 1\}$ *such that*

$$\{B_{1-k}\} \overset{\text{loose}}{\subset} \mathrm{Conv}_{\mathbb{R}^2}\big(\{B_k\} \cup (\{A_0, A_1, A_2\} \setminus \{A_j\})\big).$$

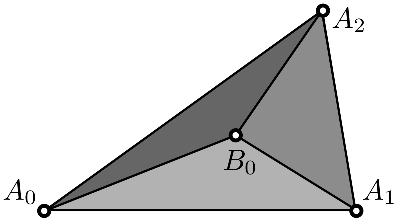

FIGURE 3. Illustration for the proof of Lemma 3.4

Proof. If B_1 is in the interior of one of the three little triangles that are colored with different shades of grey in Figure 3, then we can let $k := 0$. Otherwise, B_1 is an internal point of one of the line segments $[A_0, B_0]$, $[A_1, B_0]$, and $[A_2, B_0]$, and we can let $k := 1$. In both cases, it is clear that we can choose an appropriate $j \in \{0, 1, 2\}$. □

Lemma 3.5. *Condition 1.1(c), even without assuming 3.2(d), implies (1.2), that is, the conclusion of Lemma 3.2.*

Proof. Since $\mathcal{U}_0$ and $\mathcal{U}_1$ play symmetric roles, we can assume that $\mathcal{U}_0 = \{B_0\}$ is a singleton. We can assume also that $B_0 \in \mathrm{Int}(\triangle_{A_0,A_1,A_2})$, because otherwise the statement is trivial. If there exists a point $B_1 \in \mathcal{U}_1$ and a subscript $j \in \{0, 1, 2\}$ such that

$$B_0 \in \mathrm{Conv}_{\mathbb{R}^2}\big(\{B_1\} \cup (\{A_0, A_1, A_2\} \setminus \{A_j\})\big), \tag{3.9}$$ ⟨13⟩

then (1.2) holds with $k = 1$ and this j. So, we assume that (3.9) fails for all $B_1 \in \mathcal{U}_1$ and all $j \in \{0, 1, 2\}$. Then, by Lemma 3.4 if B_1 below is in $\mathrm{Int}(\triangle_{A_0,A_1,A_2})$ or trivially if $B_1 \in \partial\triangle_{A_0,A_1,A_2}$,

$$\begin{cases} \text{for each } B_1 \in \mathcal{U}_1, \text{ there is a smallest } j = j(B_1) \in \{0,1,2\} \text{ such that} \\ B_1 \in \mathrm{Conv}_{\mathbb{R}^2}\big(\{B_0\} \cup (\{A_0, A_1, A_2\} \setminus \{A_{j(B_1)}\})\big). \end{cases} \tag{3.10}$$ ⟨13⟩

If $j = j(B_1)$ does not depend on $B_1 \in \mathcal{U}_1$, then (3.10) gives the satisfaction of (1.2) with $k = 0$ and this j. For the sake of contradiction, suppose that $j(B_1)$ depends on $B_1 \in \mathcal{U}_1$. By (3.10), this means that there are points B_1' and B_1'' in $\mathcal{U}_1$ that belong to distinct little triangles (colored by different shades of grey) in Figure 3. By convexity, $[B_1', B_1''] \subseteq \mathcal{U}_1$. Hence, $\mathcal{U}_1$ has a point B_1 that belongs to one of the line segments $[A_0, B_0]$, $[A_1, B_0]$, and $[A_2, B_0]$. This B_1 shows the validity of (3.9) for some j, which contradicts our assumption that (3.9) fails for all j. Thus, $j(B_1)$ does not depend on $B_1 \in \mathcal{U}_1$, completing the proof. □

3.4. **Internally tangent edge-free compact convex sets.** We say that $\mathcal{U}_0$ and $\mathcal{U}_1$ subject to 1.1(a) or 1.1(b) are *internally tangent* if they have a common pointed supporting line. For example, as it is shown in Figure 4, if

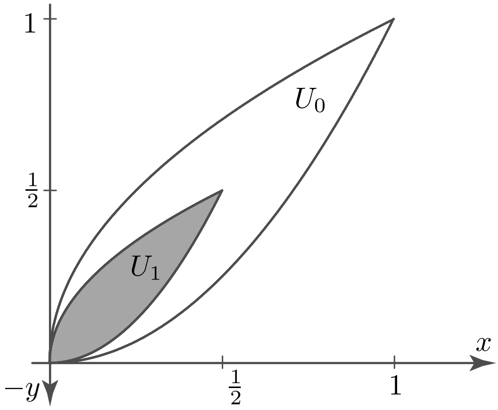

FIGURE 4. Two internally tangent edge-free compact convex sets

$$\begin{aligned}&\mathcal{U}_0 := \{\langle x, y\rangle : 0 \le x \le 1,\ \ x^2 \le y \le 1-(x-1)^2\} \text{ and}\\ &\mathcal{U}_1 := \boldsymbol{\chi}_{\langle 0,0\rangle,1/2}(\mathcal{U}_0),\end{aligned} \tag{3.11}$$ ⟨14⟩

then $\mathcal{U}_0$ and $\mathcal{U}_1$ are internally tangent edge-free compact convex sets. Let $O = \langle 0, 0\rangle$. Denoting the abscissa axis with the usual orientation $\langle 1, 0\rangle \in \mathcal{C}_{\text{unit}}$ and the ordinate axis with the unusual reverse orientation $\langle 0, -1\rangle \in \mathcal{C}_{\text{unit}}$ by x and $-y$, respectively, both $\langle O, x\rangle$ and $\langle O, -y\rangle$ are common pointed supporting lines of $\mathcal{U}_0$ and $\mathcal{U}_1$. This shows that condition 1.1(a) together with 3.2(d) do not imply the uniqueness of the common supporting lines through a point of $\partial\mathcal{U}_0 \cap \partial\mathcal{U}_1$ if $\mathcal{U}_0$ and $\mathcal{U}_1$ are internally tangent. In case of (3.11) and similar cases, these pointed supporting lines have the same support point and $\mathcal{U}_0$ and $\mathcal{U}_1$ are tangent to each other in some sense. The aim of this subsection is to prove the following lemma.

Lemma 3.6. *If $\mathcal{U}_0$ and $\mathcal{U}_1$ are non-singleton, internally tangent, edge-free compact convex subsets of $\mathbb{R}^2$, then the following two assertions hold.*

(i) *If $\mathcal{U}_1 = \boldsymbol{\chi}_{P,\lambda}(\mathcal{U}_0)$ for some $0 < \lambda \in \mathbb{R}$ and $P \in \mathbb{R}^2$, as in 1.1(a), then either $\mathcal{U}_1 = \mathcal{U}_0$ and $\lambda = 1$, or $\lambda \ne 1$ and $\partial\mathcal{U}_0 \cap \partial\mathcal{U}_1 = \{P\}$. Furthermore, if $\lambda > 1$ then $\mathcal{U}_1 \supseteq \mathcal{U}_0$ while $0 < \lambda < 1$ implies that $\mathcal{U}_1 \subseteq \mathcal{U}_0$.*

(ii) *If $\mathcal{U}_1$ is obtained from $\mathcal{U}_0$ by a translation as in 1.1(b), then $\mathcal{U}_1 = \mathcal{U}_0$ and the translation in question is the identity map.*

Note that this lemma fails without assuming that $\mathcal{U}_0$ and $\mathcal{U}_1$ are edge-free. To exemplify this, let $\mathcal{U}_0$ be the rectangle $\{\langle x, y\rangle : -2 \le x \le 2 \text{ and } 0 \le y \le 2\}$. Then $\mathcal{U}_1 := \boldsymbol{\chi}_{\langle 4,2\rangle,1/2}(\mathcal{U}_0)$ and $\mathcal{U}_1 := \{\langle x+1, y\rangle : \langle x, y\rangle \in \mathcal{U}_0\}$ would witness the failure of 3.6(i) and that of 3.6(ii), respectively.

Proof. Let $\langle P^*, \ell^*\rangle$ be a common pointed supporting line of $\mathcal{U}_0$ and $\mathcal{U}_1$.

First, assume that $\mathcal{U}_1 = \boldsymbol{\chi}_{P,\lambda}(\mathcal{U}_0)$ as in (i). We can assume that $\mathcal{U}_0 \neq \mathcal{U}_1$ since otherwise the lemma is trivial. So we know that $0 < \lambda \neq 1$. Since $\langle P^*, \ell^*\rangle$ is a pointed supporting line of $\mathcal{U}_0$, so is $\langle P', \ell'\rangle := \langle \boldsymbol{\chi}_{P,\lambda}(P^*), \boldsymbol{\chi}_{P,\lambda}(\ell^*)\rangle$ of $\mathcal{U}_1 = \boldsymbol{\chi}_{P,\lambda}(\mathcal{U}_0)$. We have that $\mathrm{dir}(\ell') = \mathrm{dir}(\ell^*)$; note that this is one of the reasons that $\lambda > 0$ is always assumed in this paper. It is well known from the folklore that for each $\alpha \in \mathcal{C}_{\mathrm{unit}}$ and every compact convex set $\mathcal{U}$,

$$\mathcal{U} \text{ has exactly one directed supporting line of direction } \alpha; \tag{3.12}$$ ⟨22⟩

see Bonnesen and Fenchel [7], Yaglom and Boltyanskiĭ [60, page 8], or Czédli and Stachó [28]. Hence, ℓ' and ℓ^* are the same supporting lines of $\mathcal{U}_1$. Since $\mathcal{U}_1$ is edge-free, $\ell^* = \ell'$ has only one support point, whence $P^* = P'$. So $P^* = P' = \boldsymbol{\chi}_{P,\lambda}(P^*)$. Since $\lambda \neq 1$, the homothety $\boldsymbol{\chi}_{P,\lambda}$ has only one fixed point, whereby $P^* = P$, as required. Next, let Q be an arbitrary element of $\partial\mathcal{U}_0 \cap \partial\mathcal{U}_1$. For the sake of contradiction, suppose that $Q \neq P$. Since $\lambda \neq 1$, the *collinear* points P, Q, and $Q' := \boldsymbol{\chi}_{P,\lambda}(Q)$ are pairwise distinct. Since the $\boldsymbol{\chi}_{P,\lambda}$-image of a boundary point is a boundary point, these three collinear points belong to $\partial\mathcal{U}_1$. This contradicts Lemma 3.1 and settles the first sentence of (i).

Since $\boldsymbol{\chi}_{P,1/\lambda}$ is the inverse of $\boldsymbol{\chi}_{P,\lambda}$, it suffices to prove the second sentence of (i) only for $\lambda > 1$, because then the case $0 < \lambda < 1$ will follow by replacing $\langle \mathcal{U}_0, \mathcal{U}_1, \lambda\rangle$ by $\langle \mathcal{U}_1, \mathcal{U}_0, 1/\lambda\rangle$. So, let X be an arbitrary point of $\mathcal{U}_0$. Using that $X \in \mathrm{Conv}_{\mathbb{R}^2}(\{P, \boldsymbol{\chi}_{P,\lambda}(X)\})$, $P = P^* = P' \in \mathcal{U}_1$, and $\boldsymbol{\chi}_{P,\lambda}(X) \in \boldsymbol{\chi}_{P,\lambda}(\mathcal{U}_0) = \mathcal{U}_1$, the convexity of $\mathcal{U}_1$ implies that $X \in \mathcal{U}_1$, as required. This completes the proof of part (i).

The argument for (ii) is similar. Let φ denote the translation such that $\mathcal{U}_1 = \varphi(\mathcal{U}_0)$. Let $\langle P', \ell'\rangle := \langle \varphi(P^*), \varphi(\ell^*)\rangle$. As in the previous paragraph, we obtain that $\langle P', \ell'\rangle$ and $\langle P^*, \ell^*\rangle$ are both supporting lines of $\mathcal{U}_1$. Since $\mathrm{dir}(\ell') = \mathrm{dir}(\ell^*)$, we have that $\ell' = \ell^*$. Thus, using that $\mathcal{U}_1$ is edge-free, we obtain that $P' = P^*$. So $P' = \varphi(P^*)$ is a fixed point of the translation φ. Hence, φ is the identity map, and we conclude that $\mathcal{U}_1 = \varphi(\mathcal{U}_0) = \mathcal{U}_0$, as required. □

3.5. **Technical lemmas.** We compose maps from right to left, so note the rule $(\varphi \circ \psi)(x) = \varphi\big(\psi(x)\big)$. The first technical lemma we need is the following.

Lemma 3.7. *Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homothety $\boldsymbol{\chi}_{P,\lambda}$ or a translation, let $P_0 \in \mathbb{R}^2$ be a point, and let $P_1 = \varphi(P_0)$. Then for every $\xi \in \mathbb{R} \setminus \{0\}$,*

$$\varphi \circ \boldsymbol{\chi}_{P_0,\xi} = \boldsymbol{\chi}_{P_1,\xi} \circ \varphi \quad \textit{or, equivalently,} \quad \boldsymbol{\chi}_{P_1,\xi} = \varphi \circ \boldsymbol{\chi}_{P_0,\xi} \circ \varphi^{-1}.$$

Proof. Since $\varphi \circ \boldsymbol{\chi}_{P_0,\xi} \circ \varphi^{-1}$ is clearly a homothety of ratio ξ that fixes P_1, this homothety is $\boldsymbol{\chi}_{P_1,\xi}$, as required. □

The next technical lemma will also be needed. It follows by straightforward computation with the help of computer algebra; an appropriate worksheet for Maple

V Release 5 is available from the homepage of the first author. After stating the lemma, we give a more geometrical and *short* proof.

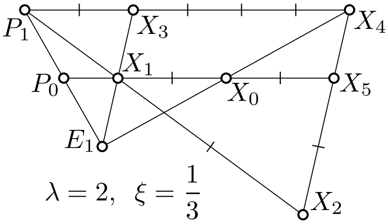

FIGURE 5. An illustration of Lemma 3.8

Lemma 3.8. *Let $\lambda, \xi \in \mathbb{R} \setminus \{0\}$, let $E_1, P_0, X_0 \in \mathbb{R}^2$, and define the points*

(1) $P_1 := \boldsymbol{\chi}_{E_1,\lambda}(P_0)$, (2) $X_1 := \boldsymbol{\chi}_{P_0,\xi}(X_0)$, (3) $X_2 := \boldsymbol{\chi}_{P_1,1/\xi}(X_1)$,
(4) $X_3 := \boldsymbol{\chi}_{E_1,\lambda}(X_1)$, (5) $X_4 := \boldsymbol{\chi}_{P_1,1/\xi}(X_3)$ (6) $X_5 := \boldsymbol{\chi}_{X_4,\xi}(X_2)$.

Then $\boldsymbol{\chi}_{X_1,1/\lambda}(X_5) = X_0$.

Proof. If $\xi = 1$ or $\lambda = 1$, then the statement is obvious. If $\xi \neq 1 \neq \lambda$, then Figure 5 visualizes what we have. By (3) and (5) we deduce $\xi\overrightarrow{X_4X_2} = \overrightarrow{X_3X_1}$, hence (6) gives $\overrightarrow{X_4X_5} = \overrightarrow{X_3X_1}$, so quadrangle $\langle X_1, X_3, X_4, X_5\rangle$ is a parallelogram. Thus,

$$\overrightarrow{X_1X_5} = \overrightarrow{X_3X_4} \overset{(5)}{=} \frac{\xi - 1}{\xi}\overrightarrow{X_3P_1} \overset{(1),(4)}{=} \frac{\lambda(\xi - 1)}{\xi}\overrightarrow{X_1P_0} \overset{(2)}{=} \lambda\overrightarrow{X_1X_0}. \qquad \square$$

The following lemma is well known, especially without the adjective "positive". However, there are other variants and the corresponding terminology is not unique in the literature; for example, Schneider [57, page xii] includes translations in the concept of positive homotheties. The terminological ambiguity in the literature justifies that we formulate this lemma and give its trivial proof.

Lemma 3.9. *Let G be the collection of all positive homotheties and all translations of the plane. Then G is a group with respect to composition.*

Proof. It suffices to show that if φ_1 and φ_2 belongs to G, then so does $\varphi := \varphi_1 \circ \varphi_2$. Since φ_1 and φ_2 are similarity transformations that preserve the directions of directed lines, the same holds for φ. This implies that $\varphi \in G$. $\square$

3.6. **The lion's share of the proof.** First, we prove the following lemma.

Lemma 3.10. *Assume that $\mathcal{U}_0$ is an edge-free compact convex subset of $\mathbb{R}^2$, $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a positive homothety or a translation, $\mathcal{U}_1 = \varphi(\mathcal{U}_0)$, $P_0 \in \mathrm{Int}(\mathcal{U}_0)$, $P_1 = \varphi(P_0)$, $\xi \in (0,1) = [0,1] \setminus \{0,1\}$, $\mathcal{U}_0(\xi) = \boldsymbol{\chi}_{P_0,\xi}(\mathcal{U}_0)$, $\mathcal{U}_1(\xi) = \boldsymbol{\chi}_{P_1,\xi}(\mathcal{U}_1)$, and $\mathcal{U}_0(\xi)$ and $\mathcal{U}_1(\xi)$ are internally tangent. Then at least one of the following two assertions hold:*

(i) $\mathcal{U}_0 \subseteq \mathcal{U}_1$ *and* $\mathcal{U}_0(\xi) \subseteq \mathcal{U}_1(\xi)$, *or*
(ii) $\mathcal{U}_1 \subseteq \mathcal{U}_0$ *and* $\mathcal{U}_1(\xi) \subseteq \mathcal{U}_0(\xi)$.

Proof. We can assume that φ is not the identity map, since otherwise the statement trivially holds. Computing by Lemma 3.7, we obtain that

$$\mathcal{U}_1(\xi) = \boldsymbol{\chi}_{P_1,\xi}\big(\varphi(\mathcal{U}_0)\big) = (\boldsymbol{\chi}_{P_1,\xi} \circ \varphi \circ \boldsymbol{\chi}_{P_0,1/\xi})\big(\mathcal{U}_0(\xi)\big) \stackrel{\text{Lem. 3.7}}{=} (\varphi \circ \boldsymbol{\chi}_{P_0,\xi} \circ \boldsymbol{\chi}_{P_0,1/\xi})\big(\mathcal{U}_0(\xi)\big) = \varphi\big(\mathcal{U}_0(\xi)\big), \tag{3.13}$$ ⟨17⟩

which shows that Lemma 3.6 is applicable to the triplet $\langle \mathcal{U}_0(\xi), \mathcal{U}_1(\xi), \varphi\rangle$. Let $\langle E_1, \ell\rangle$ be a common pointed supporting line of $\mathcal{U}_0(\xi)$ and $\mathcal{U}_1(\xi)$. Since φ is not the identity map, Lemma 3.6 gives that $\varphi = \boldsymbol{\chi}_{E_1,\lambda}$ for some $\lambda > 0$. The systems $\langle \mathcal{U}_0, \mathcal{U}_1, \lambda, \varphi = \boldsymbol{\chi}_{E_1,\lambda}\rangle$ and $\langle \mathcal{U}_1, \mathcal{U}_0, 1/\lambda, \varphi^{-1} = \boldsymbol{\chi}_{E_1,1/\lambda}\rangle$ play symmetric roles, whence we can assume that $\lambda \ge 1$. We obtain by Lemma 3.6 that

$$\mathcal{U}_0(\xi) \subseteq \mathcal{U}_1(\xi). \tag{3.14}$$ ⟨17⟩

In order to prove the inclusion $\mathcal{U}_0 \subseteq \mathcal{U}_1$, let $X_0 \in \mathcal{U}_0$. Since $\varphi = \boldsymbol{\chi}_{E_1,\lambda}$, we have that $P_1 = \boldsymbol{\chi}_{E_1,\lambda}(P_0)$. Consider the points $X_1, \dots, X_5$ defined in Lemma 3.8. By the definition of $\mathcal{U}_0(\xi)$, we have that $X_1 \in \mathcal{U}_0(\xi)$, whereby (3.14) yields that $X_1 \in \mathcal{U}_1(\xi)$. Thus, since $\boldsymbol{\chi}_{P_1,1/\xi}$ is the inverse of $\boldsymbol{\chi}_{P_1,\xi}$, the definition of $\mathcal{U}_1(\xi)$ leads to $X_2 \in \mathcal{U}_1$. Since $X_1 \in \mathcal{U}_0(\xi)$, we obtain by equation (4) of Lemma 3.8, $\boldsymbol{\chi}_{E_1,\lambda} = \varphi$, and (3.13) that $X_3 \in \mathcal{U}_1(\xi)$. This gives that $X_4 \in \mathcal{U}_1$. Since $X_2, X_4 \in \mathcal{U}_1$ and $0 < \xi < 1$, the convexity of $\mathcal{U}_1$ implies that $X_5 \in \mathcal{U}_1$. Using that $X_1 \in \mathcal{U}_1(\xi) \subseteq \mathcal{U}_1$ and that $0 < 1/\lambda \le 1$, the convexity of $\mathcal{U}_1$ gives that $\boldsymbol{\chi}_{X_1,1/\lambda}(X_5) \in \mathcal{U}_1$. Thus, $X_0 \in \mathcal{U}_1$ by Lemma 3.8, proving that $\mathcal{U}_0 \subseteq \mathcal{U}_1$, as required. □

Now, armed with the auxiliary statements proved so far, we are in the position to prove the (Main) Lemma 3.2.

Proof of Lemma 3.2. Lemma 3.5 allows us to assume that none of $\mathcal{U}_0$ and $\mathcal{U}_1$ is a singleton. For the sake of contradiction, suppose that the lemma fails. Let $\mathcal{U}_0, \mathcal{U}_1, A_0, A_1$, and A_2 witness this failure. We can assume that A_0, A_1, and A_2 is the counterclockwise list of the vertices of triangle $\triangle_{A_0,A_1,A_2}$; see (3.8). If 1.1(a) holds, then $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ will denote the transformation $\boldsymbol{\chi}_{P,\lambda}$ mentioned in 1.1(a). Similarly, if 1.1(b) holds, then $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ stands for a translation according to 1.1(b). In both cases, $\mathcal{U}_1 = \varphi(\mathcal{U}_0)$. Fix an internal point P_0 of $\mathcal{U}_0$, and let $P_1 := \varphi(P_0)$. For $i \in \{0,1\}$ and every real number $\xi \in [0,1]$, let

$$\mathcal{U}_i(\xi) := \boldsymbol{\chi}_{P_i,\xi}(\mathcal{U}_i). \tag{3.15}$$ ⟨17, 18, 19,

Note that $\boldsymbol{\chi}_{P_i,\xi}$ is a positive homothety only for $\xi > 0$ but (1.1) is meaningful also for $\lambda = 0$. In particular, $\mathcal{U}_i(0) := \boldsymbol{\chi}_{P_i,0}(\mathcal{U}_i)$ makes sense and it is understood as $\{P_i\}$. For $\xi = 0$, we trivially have that $\mathcal{U}_1(\xi) = \varphi\big(\mathcal{U}_0(\xi)\big)$. So let $\xi > 0$. Lemma 3.7 and (3.15) yield that

$$\begin{aligned}\mathcal{U}_1(\xi) &= \boldsymbol{\chi}_{P_1,\xi}(\mathcal{U}_1)\\ &= (\varphi \circ \boldsymbol{\chi}_{P_0,\xi})\big(\varphi^{-1}(\mathcal{U}_1)\big) = \varphi\big(\boldsymbol{\chi}_{P_0,\xi}(\mathcal{U}_0)\big) = \varphi(\mathcal{U}_0(\xi)).\end{aligned} \tag{3.16}$$ ⟨19⟩

Thus,

$$\mathcal{U}_1(\xi) = \varphi\big(\mathcal{U}_0(\xi)\big), \qquad \text{for all } \xi \in [0,1] \text{ and } i \in \{0,1\}. \tag{3.17}$$

Since $\mathcal{U}_i(\xi) \subseteq \mathcal{U}_i(1) = \mathcal{U}_i \subseteq \triangle_{A_0,A_1,A_2}$, we have also that $\mathcal{U}_i(\xi) \subseteq \triangle_{A_0,A_1,A_2}$.

Let H be the set of all $\eta \in [0,1]$ such that (1.2) holds for $\mathcal{U}_0(\eta)$ and $\mathcal{U}_1(\eta)$ with some j and k. Since $0 \in H$ by Lemma 3.5, or even by Lemma 3.4, $H \neq \emptyset$. Hence, H has a supremum, which we denote by ξ. It follows from Lemma 3.4 and continuity that $\xi > 0$. A standard compactness argument shows that $\xi \in H$, that is, ξ is the maximal element of H. Since a more involved, similar, but still standard argument will be given after (4.5), we do not give the details of this compactness argument here; note that the omitted details, modulo insignificant changes, are given in the extended version, arXiv:1610.02540, of Czédli [15]. Taking our indirect assumption and $\xi = \max(H)$ into account, we have that

$$0 < \xi := \max(H) < 1. \tag{3.18}$$ ⟨21⟩

Since $\xi \in H$, we can assume that the indices are chosen so that, as Figure 6 shows, $\mathcal{U}_1(\xi)$ is included in the grey-filled "curved-backed trapezoid"

$$\mathrm{Trp}(\xi) := \mathrm{Conv}_{\mathbb{R}^2}(\{A_0, A_1\} \cup \mathcal{U}_0(\xi)). \tag{3.19}$$

In Figure 6, the "back" of this trapezoid is the thick curve connecting E_0 and F_0. If $\mathcal{U}_1(\xi)$ was included in the interior of $\mathrm{Trp}(\xi)$, then there would be a (small) positive ε such that $\mathcal{U}_1(\xi+\varepsilon) \subseteq \mathrm{Trp}(\xi) \subseteq \mathrm{Trp}(\xi+\varepsilon)$ and $\xi+\varepsilon$ would belong to H, contradicting the fact that ξ is the largest element of H. Hence, $\mathcal{U}_1(\xi) \subseteq \mathrm{Trp}(\xi)$, but the intersection $\partial\mathcal{U}_1(\xi) \cap \partial\mathrm{Trp}(\xi)$ has at least one point. So we can pick a point $E_1 \in \partial\mathcal{U}_1(\xi) \cap \partial\mathrm{Trp}(\xi)$. Since $\mathcal{U}_1(\xi) \overset{\text{loose}}{\subset} \mathcal{U}_1(1) = \mathcal{U}_1 \subseteq \triangle_{A_0,A_1,A_2}$, we have that $E_1 \notin \partial\triangle_{A_0,A_1,A_2}$. Since the "left leg", that is, the straight line segment $[A_0, E_0]$, and the "right leg" $[A_1, F_0]$ of $\mathrm{Trp}(\xi)$ play symmetric roles, it suffices to consider only the following two cases: either E_1 belongs to the "back" of $\mathrm{Trp}(\xi)$, including its endpoints E_0 and F_0, or E_1 belongs to the "left leg" $[A_0, E_0]$, excluding E_0.

First, assume that E_1 belongs to the "back" of $\mathrm{Trp}(\xi)$. Then, clearly, E_1 belongs to $\partial\mathcal{U}_0(\xi)$. Since $\mathrm{Trp}(\xi)$ is a compact convex set, it has a directed supporting line ℓ through E_1. Since $\mathcal{U}_0(\xi) \subseteq \mathrm{Trp}(\xi)$ and $\mathcal{U}_1(\xi) \subseteq \mathrm{Trp}(\xi)$, both $\mathcal{U}_0(\xi)$ and $\mathcal{U}_1(\xi)$ are on the left of ℓ. Using that E_1 is in $\mathcal{U}_0(\xi) \cap \mathcal{U}_1(\xi)$, it follows that $\langle E_1, \ell\rangle$ is a common pointed supporting line of $\mathcal{U}_0(\xi)$ and $\mathcal{U}_1(\xi)$. Hence, $\mathcal{U}_0(\xi)$ and $\mathcal{U}_1(\xi)$ are internally tangent. Furthermore, it follows from Lemma 3.9 and (3.15) that $\mathcal{U}_1(\xi)$ is obtained from $\mathcal{U}_0(\xi)$ by a translation or a positive homothety. Therefore, Lemma 3.10 yields that $\mathcal{U}_0 \subseteq \mathcal{U}_1$ or $\mathcal{U}_1 \subseteq \mathcal{U}_2$. This trivially implies (1.2), contradicting the initial assumption of the proof. Thus, the first case where E_1 belongs to the back of $\mathrm{Trp}(\xi)$ has been excluded.

Second, assume that E_1 belongs to the "left leg" $[A_0, E_0]$ as illustrated in Figure 6. Since $\mathcal{U}_1(\xi) \overset{\text{loose}}{\subset} U_1 \subseteq \triangle_{A_0,A_1,A_2}$ implies that $E_1 \notin \partial\triangle_{A_0,A_1,A_2}$, we have that $E_1 \neq A_0$. We can assume that $E_1 \neq E_0$ since the opposite case has already been

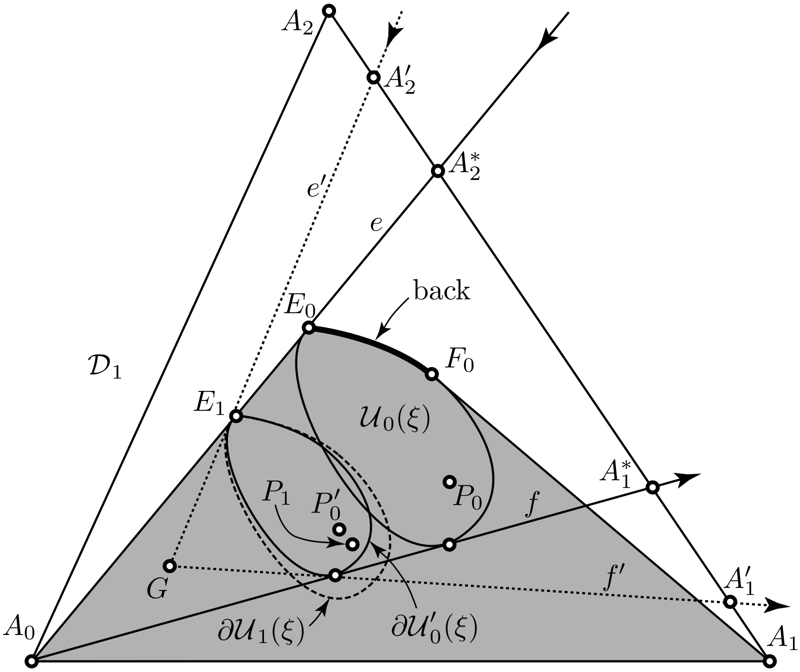

FIGURE 6. Illustration for the proof of Lemma 3.2

settled. Hence, letting $\nu := \mathrm{dist}(A_0, E_1)/\mathrm{dist}(A_0, E_0)$, we have that $0 < \nu < 1$. Let

$$\mathcal{U}_0' := \boldsymbol{\chi}_{A_0,\nu}(\mathcal{U}_0), \quad P_0' := \boldsymbol{\chi}_{A_0,\nu}(P_0), \tag{3.20}$$ ⟨19, 20⟩

$$\varphi' := \varphi \circ \boldsymbol{\chi}_{A_0,1/\nu}, \quad \text{and} \quad \mathcal{U}_0'(\xi) := \boldsymbol{\chi}_{P_0',\xi}(\mathcal{U}_0'). \tag{3.21}$$ ⟨19, 20⟩

The position of $\mathcal{U}_0'(\xi)$ in Figure 6 is justified by

$$\begin{aligned} \boldsymbol{\chi}_{A_0,\nu}(\mathcal{U}_0(\xi)) &\overset{(3.15)}{=} (\boldsymbol{\chi}_{A_0,\nu} \circ \boldsymbol{\chi}_{P_0,\xi})(\mathcal{U}_0) \\ &\overset{\text{Lem. 3.7}}{=} (\boldsymbol{\chi}_{P_0',\xi} \circ \boldsymbol{\chi}_{A_0,\nu})(\mathcal{U}_0) \overset{(3.20)}{=} \boldsymbol{\chi}_{P_0',\xi}(\mathcal{U}_0') \overset{(3.21)}{=} \mathcal{U}_0'(\xi). \end{aligned} \tag{3.22}$$ ⟨20⟩

Since $\boldsymbol{\chi}_{A_0,1/\nu}$ is the inverse of $\boldsymbol{\chi}_{A_0,\nu}$, (3.20) and (3.21) yield that

$$P_1 = \varphi'(P_0') \quad \text{and} \quad \mathcal{U}_1 = \varphi'(\mathcal{U}_0'). \tag{3.23}$$ ⟨19, 20⟩

Computing by Lemma 3.7 as in (3.16), we obtain that

$$\begin{aligned} \mathcal{U}_1(\xi) &\overset{(3.15)}{=} \boldsymbol{\chi}_{P_1,\xi}(\mathcal{U}_1) \overset{(3.23)}{=} (\boldsymbol{\chi}_{P_1,\xi} \circ \varphi')(\mathcal{U}_0') \\ &\overset{\text{Lem. 3.7}}{=} (\varphi' \circ \boldsymbol{\chi}_{P_0',\xi})(\mathcal{U}_0') \overset{(3.21)}{=} \varphi'\big(\boldsymbol{\chi}_{P_0',\xi}(\mathcal{U}_0')\big) = \varphi'(\mathcal{U}_0'(\xi)). \end{aligned} \tag{3.24}$$ ⟨20⟩

According to Figure 6, the directed line through the "left leg" of $\mathrm{Trp}(\xi)$ will be denoted by e; clearly, $E_0, E_1, A_0 \in e$. Since $\boldsymbol{\chi}_{A_0,\nu}(E_0) = E_1$, $\boldsymbol{\chi}_{A_0,\nu}(e) = e$, and

$\chi_{A_0,\nu}$ preserves supporting lines, it follows from (3.22) that $\langle E_1, e\rangle$ is a pointed supporting line of $\mathcal{U}'_0(\xi)$. Hence, $\langle E_1, e\rangle$ is a common pointed supporting line of $\mathcal{U}'_0(\xi)$ and $\mathcal{U}_1(\xi)$. Thus, $\mathcal{U}'_0(\xi)$ and $\mathcal{U}_1(\xi)$ are internally tangent. This fact and the equalities (3.15), (3.21), (3.23), and (3.24) show that the assumptions of Lemma 3.10, with $\langle P'_0, \mathcal{U}'_0, \mathcal{U}'_0(\xi), \varphi'\rangle$ instead of $\langle P_0, \mathcal{U}_0, \mathcal{U}_0(\xi), \varphi\rangle$, hold. Hence, we obtain from Lemma 3.10 that

$$\mathcal{U}_1 \subseteq \mathcal{U}'_0 \quad \text{or} \quad \mathcal{U}'_0(\xi) \subseteq \mathcal{U}_1(\xi). \tag{3.25}$$ ⟨20⟩

Now, assume the first inclusion in (3.25). Since $0 < \nu < 1$ in the definition of $\mathcal{U}'_0$ in (3.20), we have that $\mathcal{U}'_0 \subseteq \mathrm{Conv}_{\mathbb{R}^2}(\{A_0\} \cup \mathcal{U}_0)$. This together with $\mathcal{U}_1 \subseteq \mathcal{U}'_0$ yield that $\mathcal{U}_1 \subseteq \mathrm{Conv}_{\mathbb{R}^2}(\{A_0\} \cup \mathcal{U}_0)$, whence the third line of (1.2) holds. This contradicts our indirect assumption and so excludes the first inclusion in (3.25).

So we are left with the second inclusion given in (3.25), that is, $\mathcal{U}'_0(\xi) \subseteq \mathcal{U}_1(\xi)$, as shown in Figure 6. Let $f \neq e$ be the other supporting line of $\mathcal{U}_0(\xi)$ through A_0; see Figure 6 again. It follows from (3.22) that f is a supporting line of $\mathcal{U}'_0(\xi)$ as well. The intersection of e and f with the line segment $[A_1, A_2]$ will be denoted by A^*_2 and A^*_1, respectively; see Figure 6. Since $\mathcal{U}_0(\xi) \overset{\text{loose}}{\subset} \triangle_{A_0,A_1,A_2}$, the points A^*_1 and A^*_2 are strictly between A_1 and A_2 in the line segment $[A_1, A_2]$. Using that $\xi < 1$ and $\mathcal{U}'_0(\xi) \subseteq \mathcal{U}_1(\xi) \overset{\text{loose}}{\subset} \mathcal{U}_1(1) = \mathcal{U}_1 \subseteq \triangle_{A_0,A_1,A_2}$, see (3.8), we have that $\mathrm{dist}(A_0, \mathcal{U}'_0(\xi)) > 0$. Both e and f can be turned continuously around $\mathcal{U}'_0(\xi)$. The precise meaning of this continuity is given in Czédli and Stachó [28], where *pointed* supporting lines are "slide-turned". However, the edge-freeness of $\mathcal{U}'_0(\xi)$ allows us to forget about the uniquely defined support point of a pointed supporting line when we refer to [28]. So, we turn e around $\mathcal{U}'_0(\xi)$ counterclockwise sufficiently small to obtain a supporting line e' of $\mathcal{U}'_0(\xi)$. Similarly, turn f clockwise sufficiently small to obtain a supporting line f' of $\mathcal{U}'_0(\xi)$. The meaning of "sufficiently small" here is that

— the intersection point $G \in e' \cap f'$ belongs to $\mathrm{Int}(\mathrm{Conv}_{\mathbb{R}^2}(\{A_0\} \cup \mathcal{U}'_0(\xi)) \setminus \mathcal{U}'_0(\xi))$, which is possible since $\mathrm{dist}(A_0, \mathcal{U}'_0(\xi)) > 0$, and, in addition,
— the intersection points $A'_1 \in f' \cap [A_1, A_2]$ and $A'_2 \in e' \cap [A_1, A_2]$ exist and they are strictly between A_1 and A^*_1 and A^*_2 and A_2, respectively.

By Lemma 3.3, we have that $\mathrm{Comet}(A_0, \mathcal{U}_0(\xi)) \overset{\text{loose}}{\subset} \mathrm{Comet}(G, \mathcal{U}'_0(\xi))$. This fact together with $\mathcal{U}_0(\xi) \subseteq \mathrm{Comet}(A_0, \mathcal{U}_0(\xi))$ yield $\mathcal{U}_0(\xi) \overset{\text{loose}}{\subset} \mathrm{Comet}(G, \mathcal{U}'_0(\xi))$. Since we also have that $\mathcal{U}_0(\xi) \overset{\text{loose}}{\subset} \triangle_{A_0,A_1,A_2}$ and the inclusion $\mathrm{Int}(\mathcal{V}_1) \cap \mathrm{Int}(\mathcal{V}_2) \subseteq \mathrm{Int}(\mathcal{V}_1 \cap \mathcal{V}_2)$ trivially holds for all $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathbb{R}^2$, we obtain that

$$\begin{aligned}\mathcal{U}_0(\xi) &\overset{\text{loose}}{\subset} \mathrm{Comet}(G, \mathcal{U}'_0(\xi)) \cap \triangle_{A_0,A_1,A_2}\\ &= \mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U}'_0(\xi) \cup \{A'_1, A'_2\}) \subseteq \mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U}_1(\xi) \cup \{A_1, A_2\}).\end{aligned}$$

This loose inclusion and (3.7) yield a (small) positive $\delta \in \mathbb{R}$ such that $\xi + \delta < 1$ and

$$\mathcal{U}_0(\xi+\delta) = \boldsymbol{\chi}_{P_0,(\xi+\delta)/\xi}(\mathcal{U}_0) \overset{\text{loose}}{\subset} \mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U}_1(\xi) \cup \{A_1, A_2\})$$
$$\subseteq \mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U}_1(\xi+\delta) \cup \{A_1, A_2\})$$

Hence, $\xi+\delta \in H$, contradicting $\xi = \max H$; see (3.18). This contradiction excludes the second case where E_1 belongs to the "left leg" of $\mathrm{Trp}(\xi)$.

Finally, it is a contradiction that both cases have been excluded. This completes the proof of Lemma 3.2 □

4. Getting rid of edges

Recall that *disks* are convex hulls of circles and *circles* are boundaries of disks.

Lemma 4.1. *The intersection of finitely many disks of the plane is an edge-free compact convex set whenever it is not empty.*

Proof. Let $\mathcal{D}_1, \dots, \mathcal{D}_n$ be disks such that $\mathcal{U} := \mathcal{D}_1 \cap \dots \cap \mathcal{D}_n \neq \emptyset$. Clearly, $\mathcal{U}$ is compact and convex. For the sake of contradiction, suppose that $\mathcal{U}$ is not edge-free. Then we can pick a supporting line ℓ and $2n+1$ distinct points $P_1, \dots, P_{2n+1}$ belonging to $\ell \cap \partial\mathcal{U}$. Since $\mathrm{Int}(\mathcal{D}_1) \cap \dots \cap \mathrm{Int}(\mathcal{D}_1) \subseteq \mathrm{Int}(\mathcal{U})$, none of the points P_i belongs to this intersection. Hence, for each $i \in \{1, \dots, 2n+1\}$, there is a $j = j(i)$ in $\{1, \dots, n\}$ such that $P_i \notin \mathrm{Int}(\mathcal{D}_{j(i)})$. But $P_i \in \mathcal{U} \subseteq \mathcal{D}_{j(i)}$, whence $P_i \in \ell \cap \partial\mathcal{D}_{j(i)}$. By the pigeonhole principle, there are pairwise distinct $i_1, i_2, i_3 \in \{1, \dots 2n+1\}$ such that $j(i_1) = j(i_2) = j(i_3)$. Letting j be this common value, $\{P_{i_1}, P_{i_2}, P_{i_3}\} \subseteq \ell \cap \partial\mathcal{D}_j$. This is a contradiction, because a line and a circle can have at most two points in common. □

For a positive $d \in \mathbb{R}$ and a compact convex subset $\mathcal{U}$ of $\mathbb{R}^2$, we define the *open extension* $\mathrm{OpExt}(\mathcal{U}, d)$ of $\mathcal{U}$ by d as

$$\begin{aligned}\mathrm{OpExt}(\mathcal{U}, d) &:= \{X \in \mathbb{R}^2 : (\exists Y \in \mathcal{U})\,(\mathrm{dist}(X, Y) < d)\} \\ &= \{X \in \mathbb{R}^2 : \mathrm{dist}(\{X\}, \mathcal{U}) < d)\};\end{aligned} \tag{4.1}$$

the second equality above is a consequence of the compactness of $\mathcal{U}$. Clearly, $\mathrm{OpExt}(\mathcal{U}, d)$ is an open set. For convex compact sets $\mathcal{U}, \mathcal{V} \subseteq \mathbb{R}^2$ such that $\mathcal{U} \subseteq \mathcal{V}$, we define the *abundance* of $\mathcal{V}$ over $\mathcal{U}$ as

$$\mathrm{Abd}(\mathcal{U}, \mathcal{V}) := \inf\{d \in \mathbb{R} : 0 < d \text{ and } \mathcal{V} \subseteq \mathrm{OpExt}(\mathcal{U}, d)\}. \tag{4.2}$$

In order to reduce the general case of Theorem 1.1 to the edge-free case covered by Lemma 3.2, we are going to prove the following lemma.

Lemma 4.2. *For each nonempty convex compact subset $\mathcal{U}$ of the plane $\mathbb{R}^2$, there exists a sequence $(\mathcal{U}_n \colon n \in \mathbb{N}^+)$ of* edge-free *convex compact subsets such that*

- (A) $\mathcal{U} \subseteq \mathcal{U}_{n+1} \subseteq \mathcal{U}_n$ *for all* $n \in \mathbb{N}^+$,
- (B) $\lim_{n\to\infty} \mathrm{Abd}(\mathcal{U}, \mathcal{U}_n) = 0$, *and*
- (C) $\mathcal{U} = \bigcap_{n\in\mathbb{N}^+} \mathcal{U}_n$.

Proof. A disk $\{\langle x, y\rangle : (x-a)^2 + (y-b)^2 \le r^2\}$ will be called *rational* if a, b, and r are rational numbers. By basic cardinal arithmetics, there are only countably many rational disks. Hence, there exists a sequence $(\mathcal{D}_n : n \in \mathbb{N}^+)$ consisting of all rational disks that include $\mathcal{U}$ as a subset. For $n \in \mathbb{N}^+$, let $\mathcal{U}_n := \mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n$; it is an edge-free compact convex set by Lemma 4.1, and part (A) of the lemma clearly holds. We can assume that $\mathcal{U} \subseteq \operatorname{Int}(\mathcal{D}_1)$, because otherwise we can interchange $\mathcal{D}_1$ with a much larger disk of the sequence. In Figure 7, to save space, only two arcs of $\partial\mathcal{D}_1$ are given.

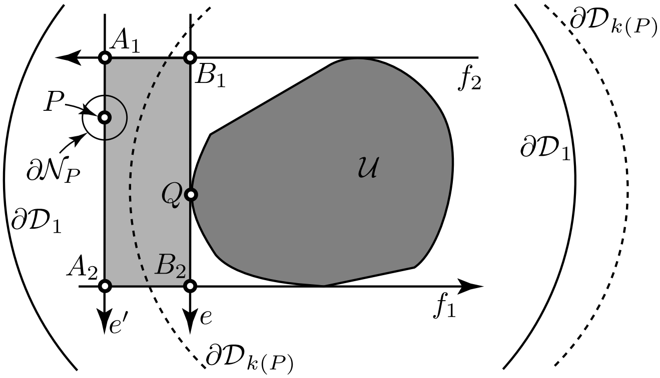

FIGURE 7. Illustration for the proof of Lemma 4.2

For an arbitrary (small) $0 < \varepsilon \in \mathbb{R}$, let $\mathcal{F}(\varepsilon) := \mathcal{D}_1 \setminus \operatorname{OpExt}(\mathcal{U}, \varepsilon)$. Clearly, $\mathcal{F}(\varepsilon)$ is a compact set. Let $P \in \mathcal{F}(\varepsilon)$. Since $P \notin \mathcal{U}$, (3.4) allows us to pick a directed supporting line e of $\mathcal{U}$ such that P is strictly on the right of e; see Figure 7. Let e' be the directed line through P such that $\operatorname{dir}(e') = \operatorname{dir}(e)$. By (3.12), there are exactly two supporting lines, f_1 and f_2, that are perpendicular to e; their role together with the light-grey rectangle in Figure 7 is to show how to choose a rational disk (with sufficiently large radius) containing $\mathcal{U}$ such that each point of this disk is strictly on the left of e'. Since this disk belongs to our sequence, it is $\mathcal{D}_{k(P)}$ for some $k(P) \in \mathbb{N}^+$; two arcs of $\partial\mathcal{D}_{k(P)}$ are given by dashed curves in the figure. Note that $P \in e'$ need not be between A_1 and A_2. It is clear by the choice of $\mathcal{D}_{k(P)}$ that we can pick a circular neighborhood $\mathcal{N}_P$ of P such that $\mathcal{N}_P \cap \mathcal{D}_{k(P)} = \emptyset$ and so $\mathcal{N}_P \cap \mathcal{U}_{k(P)} = \emptyset$. Note that $\mathcal{N}_P$ is an open set; its boundary is indicated in Figure 7. Since the collection $\{\mathcal{N}_P : P \in \mathcal{F}(\varepsilon)\}$ covers the compact set $\mathcal{F}(\varepsilon)$, we can select finitely many points $P_1, \dots, P_t$ of $\mathcal{F}(\varepsilon)$ such that $\mathcal{F}(\varepsilon) \subseteq \mathcal{N}_{P_1} \cup \cdots \cup \mathcal{N}_{P_t}$. Let $m(\varepsilon) := \max\{k(P_1), \dots, k(P_t)\}$.

Now assume that $n \ge m(\varepsilon)$ and $P \in \mathcal{F}(\varepsilon)$. Then $P \in \mathcal{N}_{P_i}$ for some $i \in \{1, \dots, t\}$, whence $\mathcal{N}_{P_i} \cap \mathcal{U}_{k(P_i)} = \emptyset$ yields that $P \notin \mathcal{U}_{k(P_i)}$. Thus, for all $n \ge m(\varepsilon)$, $P \notin \mathcal{U}_n$ since the sequence $(\mathcal{U}_n : n \in \mathbb{N}^+)$ is decreasing. This fact together with $\mathcal{U}_n \subseteq \mathcal{D}_1$ imply that, for all $n \ge m(\varepsilon)$, we have that $\mathcal{U}_n \subseteq \operatorname{OpExt}(\mathcal{U}, \varepsilon)$. This means that $\operatorname{Abd}(\mathcal{U}, \mathcal{U}_n) \le \varepsilon$ for all $n \ge m(\varepsilon)$, proving part (B) of Lemma 4.2.

Finally, part (C) follows from part (B), since every point outside $\mathcal{U}$ is at a positive distance from $\mathcal{U}$. This completes the proof of of Lemma 4.2. $\square$

Now, armed with the preparatory lemmas that we have proved so far, we are in the position to prove our theorem.

Proof of Theorem 1.1. By (3.1) and Lemma 3.5, we can assume that $\mathcal{U}_0$ and $\mathcal{U}_1$ are *convex* compact sets and none of them is a singleton. Assume also that they satisfy condition 1.1(a) or condition 1.1(b), and let φ denote the transformation from 1.1(a) or 1.1(b), respectively. Then $\mathcal{U}_1 = \varphi(\mathcal{U}_0)$. Finally, assume the premise of (1.2). Let $(\mathcal{U}_{0,n} : n \in \mathbb{N}^+)$ be a sequence provided for $\mathcal{U}_0$ by Lemma 4.2 with the notational change that we write $\mathcal{U}_0$ and $\mathcal{U}_{0,n}$ instead of $\mathcal{U}$ and $\mathcal{U}_n$, respectively. Since φ preserves the validity of Lemma 4.2, the statement of this lemma holds also for $\mathcal{U}_1$ and $\mathcal{U}_{1,n} := \varphi(\mathcal{U}_{0,n})$ instead of $\mathcal{U}$ and $\mathcal{U}_n$, respectively.

Let $W = (A_0 + A_1 + A_2)/3$ be the barycenter of $\triangle_{A_0,A_1,A_2}$. For each $n \in \mathbb{N}^+$ and $i \in \{0, 1, 2\}$, let $A_i^{(n)} := \boldsymbol{\chi}_{W,(n+1)/n}(A_i)$, and let

$$\triangle^{(n)}_{A_0,A_1,A_2} := \boldsymbol{\chi}_{W,(n+1)/n}(\triangle_{A_0,A_1,A_2}) = \mathrm{Conv}_{\mathbb{R}^2}\{A_0^{(n)}, A_1^{(n)}, A_2^{(n)}\}.$$

Observe that part (B) of Lemma 4.2 allows us to pick an integer $t(n) \in \mathbb{N}^+$ for each $n \in \mathbb{N}^+$ such that $t(n) \geq n$, $\mathcal{U}_{0,t(n)} \subseteq \triangle^{(n)}_{A_0,A_1,A_2}$, and $\mathcal{U}_{1,t(n)} \subseteq \triangle^{(n)}_{A_0,A_1,A_2}$. Also, we know from Lemma 4.2 that $\mathcal{U}_{0,t(n)}$ is edge-free. Thus, Lemma 3.2 is applicable and it yields a $j(n) \in \{0, 1, 2\}$ and a $k(n) \in \{0, 1\}$ such that the inclusion in the third line of (1.2) holds with self-explanatory notational changes to be exemplified by (4.3) soon. Although $\langle j(n), k(n)\rangle \in \{0, 1, 2\} \times \{0, 1\}$ may depend on n, one of the six possible values occurs for infinitely many n. Without loss of generality, to avoid complicated notations, we can assume that $\langle j(n), k(n)\rangle$ does not depend on n, because we could work with a subsequence $(n_1, n_2, n_3, \dots)$ instead of $(1, 2, 3, \dots)$ otherwise. Also, by changing the notation if necessary, we can assume that $j(n) = 0$ and $k(n) = 0$. That is, for all $n \in \mathbb{N}^+$,

$$\mathcal{U}_{1,n} \subseteq \mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U}_{0,n} \cup \{A_1^{(n)}, A_2^{(n)}\}). \tag{4.3}$$ ⟨23⟩

As mentioned before, Lemma 4.2 and, in particular, its part (C) hold for $\mathcal{U}_1$ and $(\mathcal{U}_{1,n} : n \in \mathbb{N}^+)$. Combining this fact with (4.3), we obtain that

$$\mathcal{U}_1 \subseteq \bigcap_{n\in\mathbb{N}^+} \mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U}_{0,n} \cup \{A_1^{(n)}, A_2^{(n)}\}).$$

Hence, it suffices to show that

$$\bigcap_{n\in\mathbb{N}^+} \mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U}_{0,n} \cup \{A_1^{(n)}, A_2^{(n)}\}) \subseteq \mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U}_0 \cup \{A_1, A_2\}). \tag{4.4}$$ ⟨23, 24⟩

So assume that $P \in \mathbb{R}^2$ belongs to the intersection in (4.4). So P belongs to each of the sets we intersect in (4.4). Hence, applying Carathéodory's well-known theorem, see, for example, Schneider [57, Theorem 1.1.4], and using the convexity of $\mathcal{U}_0$, we can pick a point $X_n \in \mathcal{U}_{0,n}$ such that P is of the form

$$P = \lambda_{0,n} \cdot X_n + \lambda_{1,n} \cdot A_1^{(n)} + \lambda_{2,n} \cdot A_2^{(n)}, \tag{4.5}$$ ⟨18, 24⟩

where $\vec{\lambda}_n := \langle \lambda_{0,n}, \lambda_{1,n}, \lambda_{2,n} \rangle \in [0,1]^3$ such that $\lambda_{0,n} + \lambda_{1,n} + \lambda_{2,n} = 1$. This condition on $\vec{\lambda}_n$ means that $\vec{\lambda}_n$ belongs to the equilateral triangle

$$E := \mathrm{Conv}_{\mathbb{R}^3}(\{\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle\})$$

in the 3-dimensional space $\mathbb{R}^3$. Hence, as $(\mathcal{U}_{0,n} : n \in \mathbb{N}^+)$ is a decreasing sequence by Lemma 4.2(A), the pair $\langle X_n, \vec{\lambda}_n \rangle$ ranges in the Cartesian product $U_{0,0} \times E$, which is a compact subset of $\mathbb{R}^2 \times E$ since E is compact and so is $\mathcal{U}_{0,0}$ by Lemma 4.2. Therefore, the sequence $(\langle X_n, \vec{\lambda}_n \rangle : n \in \mathbb{N}^+)$ has a cluster point $\langle X, \vec{\lambda} \rangle := \langle X, \langle \lambda_0, \lambda_1, \lambda_2 \rangle \rangle \in \mathbb{R}^2 \times E$. So this sequence has a subsequence converging to $\langle X, \vec{\lambda} \rangle$. To simplify the notation again, we can assume that this subsequence is the whole sequence; without this assumption, the argument is similar but needs more complicated notations. Forming the limit of (4.5) and using that $\lim_{n\to\infty} A_i^{(n)} = A_i$ for $i \in \{1,2\}$, we obtain that

$$P = \lambda_0 \cdot X + \lambda_1 \cdot A_1 + \lambda_2 \cdot A_2. \tag{4.6}$$ ⟨24⟩

Since E is a compact set, it contains $\vec{\lambda} = \langle \lambda_0, \lambda_1, \lambda_2 \rangle$, that is, we have a convex linear combination in (4.6). It follows easily from Lemma 4.2(B) that $X = \lim_{n\to\infty} X_n \in \mathcal{U}_0$. These two facts and (4.6) imply that $P \in \mathrm{Conv}_{\mathbb{R}^2}(\mathcal{U}_0 \cup \{A_1, A_2\})$. This proves (4.4) since P was an arbitrary point in the intersection on the left of (4.4). The proof of Theorem 1.1 is complete. □

Acknowledgement

This research of the first author and that of the second author was supported by the Hungarian Research Grants KH 126581 and K 116451, respectively.

References

[1] Adaricheva, K., *Representing finite convex geometries by relatively convex sets*, European J. Combin. **37** (2014), 68–78. ⟨12⟩

[2] Adaricheva, K. and Bolat, M., *Representation of convex geometries by circles on the plane*, `https://arxiv.org/pdf/1609.00092`. ⟨2, 8, 10⟩

[3] Adaricheva, K. and Czédli, G., *Note on the description of join-distributive lattices by permutations*, Algebra Universalis **72** (2014), 155–162. ⟨5, 7⟩

[4] Adaricheva, K. V, Gorbunov, V. A., and Tumanov, V. I., *Join semidistributive lattices and convex geometries*, Adv. Math. **173** (2003), 1–49. ⟨6⟩

[5] Adaricheva, K. and Nation, J. B., *Convex geometries*, in "Lattice Theory: Special Topics and Applications", vol. 2, G. Grätzer and F. Wehrung, eds., Birkhäuser, 2015. ⟨5⟩

[6] Bogart, K. P., Freese, R., and Kung, J. P. S. (editors), "The Dilworth Theorems. Selected papers of Robert P. Dilworth", Birkhäuser, Boston, 1990. ⟨4⟩

[7] Bonnesen, T. and Fenchel, W., "Theory of convex bodies", Translated from the German and edited by L. Boron, C. Christenson, and B. Smith, BCS Associates, Moscow, ID, 1987. ⟨10, 15⟩
[8] Czédli, G., *The matrix of a slim semimodular lattice*, Order **29** (2012), 85–103. ⟨5⟩
[9] Czédli, G., *Representing homomorphisms of distributive lattices as restrictions of congruences of rectangular lattices*, Algebra Universalis **67** (2012), 313–345. ⟨5⟩
[10] Czédli, G., *Coordinatization of join-distributive lattices*, Algebra Universalis **71** (2014), 385–404. ⟨5, 6⟩
[11] Czédli, G., *Finite convex geometries of circles*, Discrete Math. **330** (2014) , 61–75. ⟨7⟩
[12] Czédli, G., *Patch extensions and trajectory colorings of slim rectangular lattices*, Algebra Universalis **72** (2014) 125–154. ⟨5⟩
[13] Czédli, G., *A note on congruence lattices of slim semimodular lattices*, Algebra Universalis **72** (2014), 225–230. ⟨5⟩
[14] Czédli, G., *Characterizing circles by a convex combinatorial property*, Acta Sci. Math. (Szeged) **83** (2017), 683–701. ⟨2, 8, 10⟩
[15] Czédli, G., *An easy way to a theorem of Kira Adaricheva and Madina Bolat on convexity and circles*, Acta Sci. Math. (Szeged) **83** (2017), 703–712. ⟨2, 10, 18⟩
[16] Czédli, G., *Celebrating professor George A. Grätzer*, Categories and General Algebraic Structures with Applications, http://cgasa.sbu.ac.ir/data/cgasa/news/Gratzer.pdf. ⟨1, 3⟩
[17] Czédli, G., *An interview with George A. Grätzer*, Categories and General Algebraic Structures with Applications, http://cgasa.sbu.ac.ir/data/cgasa/news/czedli-gratzer_interview-2018june1.pdf. ⟨1⟩
[18] Czédli, G., *Circles and crossing planar compact convex sets*, submitted to Acta Sci. Math. (Szeged), https://arxiv.org/pdf/1802.06457. ⟨8⟩
[19] Czédli, G. and Grätzer, G., *Notes on planar semimodular lattices. VII. Resections of planar semimodular lattices*, Order **30** (2013), 847–858. ⟨5⟩
[20] Czédli, G. and Grätzer, G., *Planar semimodular lattices: structure and diagrams*, in "Lattice theory: special topics and applications", Vol. 1, Birkhäuser/Springer, Cham, 2014, 91–130. ⟨5, 6⟩
[21] Czédli, G., Grätzer, G., and Lakser, H., *Congruence structure of planar semimodular lattices: the General Swing Lemma*, Algebra Universalis (2018, online), https://doi.org/10.1007/s00012-018-0483-2. ⟨7⟩
[22] Czédli, G. and Kincses, J., *Representing convex geometries by almost-circles*, Acta Sci. Math. (Szeged) **83** (2017), 393–414. ⟨8⟩
[23] Czédli, G. and Makay, G.: *Swing lattice game and a short proof of the swing lemma for planar semimodular lattices*, Acta Sci. Math. (Szeged) **83** (2017), 13–29. ⟨5, 7⟩
[24] Czédli, G., Ozsvárt, L., and Udvari, B., *How many ways can two composition series intersect?*, Discrete Math. **312** (2012), 3523–3536. ⟨5⟩
[25] Czédli, G. and Schmidt, E. T., *The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices*, Algebra Universalis **66** (2011), 69–79. ⟨4, 7⟩
[26] Czédli, G. and Schmidt, E. T., *Slim semimodular lattices. I. A visual approach*, Order **29** (2012), 481–497. ⟨5⟩
[27] Czédli, G. and Schmidt, E. T., *Slim semimodular lattices. II. A description by patchwork systems*, Order **30** (2013), 689–721. ⟨5⟩
[28] Czédli, G. and Stachó, L. L., *A note and a short survey on supporting lines of compact convex sets in the plane*, Acta Univ. M. Belii Ser. Math. **24** (2016), 3–14. ⟨9, 10, 15, 20⟩
[29] Dilworth, R. P., *Lattices with unique irreducible decompositions*, Ann. of Math.(2) **41** (1940), 771–777. ⟨6⟩

[30] Edelman, P. H., *Meet-distributive lattices and the anti-exchange closure*, Algebra Universalis **10** (1980), 290–299. ⟨6⟩

[31] Edelman, P. H. and Jamison, R. E., *The theory of convex geometries*, Geom. Dedicata **19** (1985), 247–271. ⟨5⟩

[32] Erdős, P. and Straus, E. G., *Über eine geometrische Frage von Fejes-Tóth*, Elem. Math. **23** (1968), 11–14. ⟨8⟩

[33] Fejes-Tóth, L., *Eine Kennzeichnung des Kreises*, Elem. Math. **22** (1967), 25–27. ⟨8⟩

[34] Freese, R., Ježek, J., and Nation, J. B., "Free lattices", Mathematical Surveys and Monographs, 42, American Mathematical Society, Providence, RI, 1995. ⟨7⟩

[35] Funayama, N., Nakayama, T., *On the distributivity of a lattice of lattice-congruences*, Proc. Imp. Acad. Tokyo **18** (1942), 553–554. ⟨3⟩

[36] Grätzer, G., "The Congruences of a Finite Lattice. A Proof-by-picture Approach", Birkhäuser, Boston, 2006. ⟨4⟩

[37] Grätzer, G., *Planar semimodular lattices: congruences*, in "Lattice theory: special topics and applications", Vol. 1, Birkhäuser/Springer, Cham, 2014, 131–165. ⟨5⟩

[38] Grätzer, G., *Congruences in slim, planar, semimodular lattices: the swing lemma*, Acta Sci. Math. (Szeged) **81** (2015), 381–397. ⟨5, 7⟩

[39] Grätzer, G., *On a result of Gábor Czédli concerning congruence lattices of planar semimodular lattices*, Acta Sci. Math. (Szeged) **81** (2015), 25–32. ⟨5⟩

[40] Grätzer, G., "The Congruences of a Finite Lattice. A Proof-by-picture Approach", second edition, Birkhäuser/Springer, 2016. ⟨4, 5⟩

[41] Grätzer, G. and Knapp, E., *Notes on planar semimodular lattices. I. Construction*, Acta Sci. Math. (Szeged) **73** (2007), 445–462. ⟨3, 5, 6⟩

[42] Grätzer, G. and Knapp, E., *Notes on planar semimodular lattices. II. Congruences*, Acta Sci. Math. (Szeged) **74** (2008), 37–47. ⟨3, 6⟩

[43] Grätzer, G. and Knapp, E., *A note on planar semimodular lattices*, Algebra Universalis **58** (2008), 497–499. ⟨3, 6⟩

[44] Grätzer, G. and Knapp, E., *Notes on planar semimodular lattices. III. Congruences of rectangular lattices*, Acta Sci. Math. (Szeged) **75** (2009), 29–48. ⟨3, 4, 6⟩

[45] Grätzer, G. and Knapp, E., *Notes on planar semimodular lattices. IV. The size of a minimal congruence lattice representation with rectangular lattices*, Acta Sci. Math. (Szeged) **76** (2010), 3–26. ⟨3, 4, 6⟩

[46] Grätzer, G. and Nation, J. B., *A new look at the Jordan-Hölder theorem for semimodular lattices*, Algebra Universalis **64** (2010), 309–311. ⟨4, 6⟩

[47] Grätzer, G. and Schmidt, E. T., *On congruence lattices of lattices*, Acta Math. Acad. Sci. Hungar. **13** (1962), 179–185. ⟨4⟩

[48] Grätzer, G. and Schmidt, E. T., *An extension theorem for planar semimodular lattices*, Period. Math. Hungar. **69** (2014), 32–40. ⟨5⟩

[49] Grätzer, G. and Schmidt, E. T., *A short proof of the congruence representation theorem of rectangular lattices*, Algebra Universalis **71** (2014), 65–68. ⟨5⟩

[50] Hüsseinov, F., *A note on the closedness of the convex hull and its applications*, Journal of Convex Analysis **6** (1999), 387–393. ⟨9⟩

[51] Jónsson, B. and Nation, J. B., *A report on sublattices of a free lattice*, in "Contributions to Universal Algebra", Colloq. Math. Soc. János Bolyai, Vol. **17**, North-Holland, Amsterdam, 1977, 223–257. ⟨7⟩

[52] Kashiwabara, K., Nakamura, M., and Okamoto, Y., *The affine representation theorem for abstract convex geometries*, Comput. Geom. **30** (2005), 129–144. ⟨7⟩

[53] Kincses, J.: *On the representation of finite convex geometries with convex sets*, Acta Sci. Math. (Szeged) **83** (2017), 301–312. ⟨8⟩

[54] Latecki, L., Rosenfeld., A., and Silverman, R., *Generalized convexity: CP_3 and boundaries of convex sets*, Pattern Recognition **28** (1995), 1191–1199. ⟨9⟩

[55] Monjardet, B., *A use for frequently rediscovering a concept*, Order **1** (1985), 415–417. ⟨5⟩

[56] Richter, M. and Rogers, L. G., *Embedding convex geometries and a bound on convex dimension*, Discrete Math. **340** (2017), 1059–1063. ⟨7⟩

[57] Schneider, R., "Convex bodies: the Brunn-Minkowski theory", Encyclopedia of mathematics and its applications **44**, Cambridge University Press, Cambridge, 1993. ⟨16, 24⟩

[58] Toponogov, V. A., "Differential Geometry of Curves and Surfaces, A Concise Guide", Birkhäuser, Boston, 2006. ⟨9⟩

[59] Wehrung, F., *A solution to Dilworth's congruence lattice problem*, Adv. Math. **216** (2007), 610–625. ⟨4⟩

[60] Yaglom, I. M. and Boltyanskiĭ, V. G., "Convex Figures", English translation, Holt, Rinehart and Winston Inc., New York, 1961. ⟨15⟩

Gábor Czédli, University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, Hungary 6720

E-mail address: `czedli@math.u-szeged.hu`

URL: `http://www.math.u-szeged.hu/~czedli/`

Árpád Kurusa, University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, Hungary 6720

E-mail address: `kurusa@math.u-szeged.hu`

URL: `http://www.math.u-szeged.hu/~kurusa/`