

The action-angle dual of an integrable Hamiltonian system of Ruijsenaars–Schneider–van Diejen type

L. Fehér^{a,b} and I. Marshall^c

^aDepartment of Theoretical Physics, University of Szeged
Tisza Lajos krt 84-86, H-6720 Szeged, Hungary
e-mail: lfeher@physx.u-szeged.hu

^bDepartment of Theoretical Physics, WIGNER RCP, RMKI
H-1525 Budapest, P.O.B. 49, Hungary

^c Faculty of Mathematics, Higher School of Economics
Ulitsa Vavilova 7, Moscow, Russia
e-mail: imarshall@hse.ru

Abstract

Integrable deformations of the hyperbolic and trigonometric BC_n Sutherland models were recently derived via Hamiltonian reduction of certain free systems on the Heisenberg doubles of $SU(n, n)$ and $SU(2n)$, respectively. As a step towards constructing action-angle variables for these models, we here apply the same reduction to a different free system on the double of $SU(2n)$ and thereby obtain a novel integrable many-body model of Ruijsenaars–Schneider–van Diejen type that is in action-angle duality with the respective deformed Sutherland model.

1 Introduction

The study of integrable many-body models of Calogero–Moser–Sutherland type began with the seminal papers [1, 2, 3] and has since been enriched by several contributions, including notably the generalization to arbitrary root systems by Olshanetsky and Perelomov [4], and the discovery of relativistic deformations by Ruijsenaars and Schneider [5] developed further by van Diejen [6] and others. These models are ubiquitous in physical applications and are connected to important fields of mathematics; see the reviews [7, 8, 9, 10, 11, 12].

At the classical level, these models exhibit intriguing action-angle duality relations [13, 14]. The duality of two integrable many-body models means that the position variables of one model serve also as the action variables of the other one, and vice versa. The pioneering work of Ruijsenaars [13, 14] relied on direct methods, building on and greatly generalizing a procedure that had appeared in the Hamiltonian reduction treatment of the simplest example [15]. By now it has become widely known [16, 17] that several dual pairs of models arise by applying Hamiltonian reduction to suitable pairs of “free systems” on a higher dimensional master phase space, and, whenever available, this interpretation provides a powerful tool for the analysis of the dual pairs. The term *free system* is a loose one: a free Hamiltonian induces a complete flow, which often can be written down explicitly, and participates in a large Abelian Poisson algebra invariant under a group of symmetries.

The goal of this paper is to exhibit action-angle duality for an integrable Ruijsenaars–Schneider–van Diejen (RSvD) type model derived recently [18, 19] by Hamiltonian reduction of the Heisenberg double [20] of the Poisson Lie group $SU(2n)$. The model in question has three free parameters and is a deformation of the trigonometric BC_n Sutherland model. It can be viewed also as a singular limit of a specialization of the five-parameter deformation due to van Diejen [6]. Its derivation [19] closely followed the analogous reduction [18] of the Heisenberg double of $SU(n, n)$. The papers [18, 19] (see also [21, 22]) applied Poisson-Lie analogues of the reduction of the cotangent bundle of $SU(n, n)$ that yields the hyperbolic BC_n Sutherland model with three arbitrary coupling constants [23]. Other relevant predecessors of the present work are the paper of Pusztai [24], where the action-angle dual of the hyperbolic BC_n Sutherland model was constructed by reduction of $T^*SU(n, n)$, and its adaptation [25] to the trigonometric case.

A key ingredient of every Hamiltonian reduction is the choice of symmetry group, which in the above examples is the group $K_+ \times K_+$ with $K_+ = SU(n, n) \cap SU(2n)$. The pertinent Heisenberg doubles carry two natural $(K_+ \times K_+)$ -symmetric free systems, and the previous works investigated reductions of those systems corresponding to geodesic motion. In the present article, we analyse the same reduction of the Heisenberg double of $SU(2n)$ as in [19], but develop a new model of the reduced phase space, wherein it is the other free system whose reduction admits a many-body interpretation. In combination with the earlier results, this allows us to establish action-angle duality between the model treated in [19] and the many-body model that we obtain here. The Hamiltonians of this pair of RSvD type models are given in equations (3.59) and (4.6) below, and their duality with one another is discussed in Section 4.

As in [18], we adopt the modest aim of finding a model for a dense open subset of the reduced phase space. Full description of the complete reduced phase space will be reported in another publication. It is worth emphasizing that the investigation of the global structure of the phase space emerging from Hamiltonian reduction can be a source of rich and surprising results. An example is the study by Wilson [26] of the adelic Grassmannian related to the complexified rational Calogero–Moser system, which opened up interesting connections

between commuting KP flows and bispectral operators. It is also worth noting that a global description is necessary in order to obtain complete flows after reduction, and this can be turned around to construct natural regularizations of several systems with singularities.

Section 2 is devoted to preparations. The two families of free Hamiltonians and their Hamiltonian vector fields are characterized in Proposition 2.1, and the reduction of interest is defined in Subsection 2.3. Our main new results are summarized by Theorem 3.2 and Theorem 3.3 in Section 3. These describe Darboux coordinates on the reduced phase space in which the simplest reduced Hamiltonian descending from the second free system acquires an RSvD form. Proposition 3.1 formulates a technical result that plays a key role in our analysis. In Section 4 we exhibit the action-angle duality between the reduced system derived in [19] and the one treated here for the first time. Finally, in Appendix A the rational limit is presented of our RSvD type Hamiltonian (3.59).

2 Preliminaries

We here collect the necessary definitions and background results that will be used later. Most of these results are fairly standard and can be found in many sources (see e.g. [27]).

2.1 Group actions and invariants

Our master phase space will be $\mathcal{M} = \mathrm{SL}(2n, \mathbb{C})$, treated as a real manifold. Let $K = \mathrm{SU}(2n)$, and B the group consisting of the upper triangular elements of $\mathrm{SL}(2n, \mathbb{C})$ with real, positive diagonal entries. We shall use the notation B_n for the analogous subgroup of $\mathrm{GL}(n, \mathbb{C})$. By the procedure of Gram–Schmidt orthogonalisation, we may write any $g \in \mathrm{SL}(2n, \mathbb{C})$ in the form

$$g = k_L b_R \tag{2.1}$$

with unique $k_L \in K$ and $b_R \in B$. Equivalently, we may write, with $k_R \in K$ and $b_L \in B$,

$$g = b_L k_R. \tag{2.2}$$

For present purposes, we favour the use of the $g = k_L b_R$ decomposition and shall often drop the subscripts, denoting the components simply as $(k, b) \in K \times B$. The natural left-multiplication action of K on \mathcal{M} generates the “left-handed” action on $K \times B$ by

$$f *_L (k, b) = (fk, b), \quad f \in K. \tag{2.3}$$

The natural right-multiplication action of K on \mathcal{M} generates the “right-handed” action on $K \times B$ by

$$f *_R (k, b) = (k', b') \quad \text{with} \quad k'b' = kb f^\dagger. \tag{2.4}$$

Let us introduce the matrix¹ $I := \mathrm{diag}(\mathbf{1}_n, -\mathbf{1}_n)$ and define

$$K_+ = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n)) = \{k \in K \mid k^\dagger I k = I\}. \tag{2.5}$$

Suppose that $b \in B$ and $f \in K$. Then there exists a unique $\tilde{f} \in K$ such that $\tilde{f} b f^\dagger \in B$, and hence we get

$$f *_R (k, b) = (k \tilde{f}^\dagger, \tilde{f} b f^\dagger). \tag{2.6}$$

¹The symbol $\mathbf{1}_n$ stands for the $n \times n$ identity matrix and later *id* will stand for $\mathbf{1}_{2n}$.

Moreover, this formula restricts to K_+ , in the sense that $f \in K_+ \Leftrightarrow \tilde{f} \in K_+$. The first claim is a direct consequence of the property of universal factorisation, while the second can be checked by writing b in block form, and then looking at each component separately.

The left-handed and right-handed actions naturally engender an action of $K_+ \times K_+$, and we shall be interested in the ring of the smooth real functions on $\mathcal{M} \simeq K \times B$ which are invariant under this action. To obtain such functions, for $Herm := \{X \in \mathbb{C}^{2n \times 2n} \mid X^\dagger = X\}$ and $qHerm := \{X \in \mathbb{C}^{2n \times 2n} \mid X^\dagger = IXI\}$, we introduce the maps $\Omega : \mathcal{M} \rightarrow Herm$ and $L : \mathcal{M} \rightarrow qHerm$, defined by

$$\begin{aligned}\Omega(kb) &= bb^\dagger, \\ L(kb) &= k^\dagger I k I.\end{aligned}\tag{2.7}$$

Clearly Ω and L are invariant with respect to the left-handed action of K_+ on \mathcal{M} . With respect to the right-handed action, from (2.6),

$$\begin{aligned}\Omega(gf^\dagger) &= \tilde{f}\Omega(g)\tilde{f}^\dagger, \\ L(gf^\dagger) &= \tilde{f}L(g)\tilde{f}^\dagger.\end{aligned}\tag{2.8}$$

From this observation there follows directly that, with respect to the obvious conjugation actions of K_+ on $Herm$ and on $qHerm$,

$$\begin{aligned}\Omega^{-1}(C^\infty(Herm)^{K_+}) &\subset C^\infty(\mathcal{M})^{K_+ \times K_+}, \\ L^{-1}(C^\infty(qHerm)^{K_+}) &\subset C^\infty(\mathcal{M})^{K_+ \times K_+}.\end{aligned}\tag{2.9}$$

Having in mind our later purpose, we next introduce a mapping $w : \mathcal{M} \rightarrow \mathbb{C}^{2n}$ as follows. Let $\hat{w} \in \mathbb{C}^{2n}$, and assume that $I\hat{w} = \hat{w}$; that is

$$\hat{w} = \begin{pmatrix} \hat{v} \\ 0 \end{pmatrix}, \quad \text{for some fixed } \hat{v} \in \mathbb{C}^n.\tag{2.10}$$

Define

$$w(kb) = k^\dagger \hat{w}.\tag{2.11}$$

From (2.6) we have, with respect to the right-handed action of K_+ on \mathcal{M} ,

$$w(gf^\dagger) = \tilde{f}w(g), \quad \forall f \in K_+, \tag{2.12}$$

whilst, with respect to the left-handed action of K_+ on \mathcal{M} , we have the tautologous statement

$$w(fg) = w(g), \quad \forall f \in K_+(\hat{w}), \tag{2.13}$$

where

$$K_+(\hat{w}) = \{f \in K_+ \mid f\hat{w} = \hat{w}\}.\tag{2.14}$$

An important relation between L and w —due to the condition $I\hat{w} = \hat{w}$ —is the self-evident

$$LIw = w.\tag{2.15}$$

2.2 Poisson structure and symmetries

The group decomposition $\mathrm{SL}(2n, \mathbb{C}) \simeq K \times B$ results in the Lie algebra decomposition $\mathfrak{sl}(2n, \mathbb{C}) \simeq \mathrm{Lie}(K) + \mathrm{Lie}(B)$, and the two subalgebras $\mathfrak{k} := \mathrm{Lie}(K)$ and $\mathfrak{b} := \mathrm{Lie}(B)$ are in natural duality with one another with respect to the invariant nondegenerate inner product on $\mathfrak{g} := \mathfrak{sl}(2n, \mathbb{C})$

$$\langle X, Y \rangle = \mathrm{Im} \, \mathrm{tr} \, XY, \quad X, Y \in \mathfrak{g}. \quad (2.16)$$

Consequently, \mathcal{M} acquires the structure of Heisenberg double in the standard way [20]. That is, $C^\infty(\mathcal{M})$ carries the (non-degenerate) Poisson bracket given by

$$\{\varphi, \psi\}(g) = \langle \nabla_g \varphi, \mathcal{R} \nabla_g \psi \rangle + \langle \nabla'_g \varphi, \mathcal{R} \nabla'_g \psi \rangle, \quad (2.17)$$

using $\mathcal{R} \in \mathrm{End}(\mathfrak{g})$ provided by half the difference of two projections, $\mathcal{R} = \frac{1}{2}(P_{\mathfrak{k}} - P_{\mathfrak{b}})$, and $\nabla_g \varphi, \nabla'_g \varphi \in \mathfrak{g}$ characterized by

$$\left. \frac{d}{dt} \right|_{t=0} \varphi(e^{tX} g e^{tY}) = \langle X, \nabla_g \varphi \rangle + \langle Y, \nabla'_g \varphi \rangle, \quad \forall X, Y \in \mathfrak{g}. \quad (2.18)$$

With respect to this extra structure, the left-handed and right-handed actions of K on \mathcal{M} are Poisson actions with momentum maps $g \mapsto b_L$ and $g \mapsto b_R^{-1}$ defined by (2.1) and (2.2).

In fact, K_+ is a Poisson Lie subgroup of K and its dual group can be identified with B/N , where $N \subset B$ is the normal subgroup of matrices having the block form,

$$N := \left\{ \begin{pmatrix} \mathbf{1}_n & X \\ 0 & \mathbf{1}_n \end{pmatrix} \middle| X \in \mathbb{C}^{n \times n} \right\}. \quad (2.19)$$

Denoting the projection $B \rightarrow B/N$ by π_N , the momentum maps generating the left-handed and right-handed actions of K_+ on \mathcal{M} are respectively the maps

$$\begin{aligned} b_L k_R &= g \mapsto \pi_N(b_L), \\ k_L b_R &= g \mapsto \pi_N(b_R^{-1}). \end{aligned} \quad (2.20)$$

Proposition 2.1. *The functions F_l and Φ_l , defined by*

$$\begin{aligned} F_l(g) &= \frac{1}{2l} \mathrm{tr} \, \Omega(g)^l, \\ \Phi_l(g) &= \frac{1}{2l} \mathrm{tr} \, L(g)^l, \end{aligned} \quad l = 1, 2, \dots \quad (2.21)$$

are all invariant with respect to the action of the symmetry group $K_+ \times K_+$. They form two separate families of functions in involution on \mathcal{M} ; that is

$$\{F_{l_1}, F_{l_2}\} = 0, \quad \forall l_1, l_2, \quad (2.22)$$

and

$$\{\Phi_{l_1}, \Phi_{l_2}\} = 0, \quad \forall l_1, l_2. \quad (2.23)$$

The Hamiltonian vector field corresponding to F_l is expressed in terms of the K and B components by

$$\mathbb{X}_{F_l}(g) : \begin{cases} \dot{k} = ik [\Omega^l - \nu_l \mathrm{id}] \\ \dot{b} = 0 \end{cases}, \quad \text{with } \nu_l = (2n)^{-1} \mathrm{tr} \, \Omega^l. \quad (2.24)$$

The Hamiltonian vector field corresponding to Φ_l is expressed in terms of the K and B components by

$$\mathbb{X}_{\Phi_l}(g) : \begin{cases} \dot{k} = \frac{1}{2}ik(IL^{l-1} - L^{l+1}I - IL^l + L^lI) \\ \dot{b} = \frac{1}{2}i(id + I)L^l(id - I)b. \end{cases} \quad (2.25)$$

Each of these vector fields generates a complete flow on \mathcal{M} .

Proof. For both families, $(K_+ \times K_+)$ -invariance is obvious from (2.9), and the involutivity properties may be deduced directly from the forms of the respective Hamiltonian vector fields. The formula for \mathbb{X}_{F_l} is obtained by straightforward application of the definitions (2.17) and (2.18). The derivation for \mathbb{X}_{Φ_l} is more lengthy, proceeding via the observation that $\nabla'_g \Phi_l \in \mathfrak{b}$, which implies that $\dot{g}g^{-1} = -[\nabla'_g \Phi_l]_{\mathfrak{b}}$, and this can be written explicitly utilizing the fact that $X^\dagger = -IXI$ entails $X_{\mathfrak{b}} := P_{\mathfrak{b}}(X) = \frac{1}{2}(id + I)X(id - I)$. The completeness property of the flow of \mathbb{X}_{F_l} is plain, while for \mathbb{X}_{Φ_l} it follows by appeal to the compactness of K , using that $\dot{b}b^{-1}$ in (2.25) depends only on k . \square

It will be important for us to have the projections of \mathbb{X}_{F_l} and \mathbb{X}_{Φ_l} expressed in terms of L , Ω and w . These follow directly from (2.24) and (2.25), using (2.7) and (2.11), and are respectively given by

$$\mathbb{X}_{F_l}(g) \Rightarrow \begin{cases} \dot{L}I = [LI, i\Omega^l] \\ \dot{\Omega} = 0 \\ \dot{w} = -i[\Omega^l - \nu_l id]w \end{cases} \quad (2.26)$$

and

$$X_{\Phi_l}(g) \Rightarrow \begin{cases} \dot{L} = \frac{1}{2}i[2L^l - L^{l-1} - L^{l+1}, I] \\ \dot{\Omega} = \frac{1}{2}i(id + I)L^l(id - I)\Omega + \frac{1}{2}i\Omega(id - I)L^l(id + I) \\ \dot{w} = \frac{1}{2}i(id + I)(L^l - L^{l-1})w. \end{cases} \quad (2.27)$$

2.3 Reduction of the systems $\{F_l\}$ and $\{\Phi_l\}$

In principle, one can perform reduction by setting the diagonal $n \times n$ blocks of b_L and b_R to arbitrary constants, elements of B_n , and then projecting to the quotient of the resulting momentum constraint surface, \mathcal{M}_0 , by the isotropy subgroup in $K_+ \times K_+$ corresponding to the constraints. The quotient, the reduced phase space \mathcal{M}_{red} , is naturally a smooth symplectic manifold if standard regularity conditions are met (see e.g. [28]). The functions F_l and Φ_l then descend to smooth functions F_l^{red} and Φ_l^{red} on \mathcal{M}_{red} forming Abelian Poisson algebras with respect to the reduced symplectic structure. The isotropy group of the constraints is also known as the *gauge group*, and the associated transformations of \mathcal{M}_0 are often called *gauge transformations*.

The following result gives us a device (used already in [18, 22]) whereby the momentum constraints are expressed as explicit functions of $g \in \mathcal{M}$. The proof is a simple exercise.

Proposition 2.2. *Suppose $\mu_1, \mu_2, \tilde{\mu}_1, \tilde{\mu}_2 \in B_n$ are given. The condition*

$$\mathcal{M} \ni g = k_L b_R \quad \text{with} \quad b_R = \begin{pmatrix} \mu_1 & * \\ 0 & \mu_2 \end{pmatrix} \quad (2.28)$$

is equivalent to

$$g^\dagger g - g^\dagger g \begin{pmatrix} (\mu_1^\dagger \mu_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix} g^\dagger g = \begin{pmatrix} 0 & 0 \\ 0 & \mu_2^\dagger \mu_2 \end{pmatrix}, \quad (2.29)$$

and the condition

$$\mathcal{M} \ni g = b_L k_R \quad \text{with} \quad b_L = \begin{pmatrix} \tilde{\mu}_1 & * \\ 0 & \tilde{\mu}_2 \end{pmatrix} \quad (2.30)$$

is equivalent to

$$gg^\dagger - gg^\dagger \begin{pmatrix} 0 & 0 \\ 0 & (\tilde{\mu}_2 \tilde{\mu}_2^\dagger)^{-1} \end{pmatrix} gg^\dagger = \begin{pmatrix} \tilde{\mu}_1 \tilde{\mu}_1^\dagger & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.31)$$

In the present work, we study reduction under the following constraints. We choose real, positive numbers x, y, α , supposing additionally that $\alpha < 1$, and then fix the constraint surface \mathcal{M}_0 by

$$\mathcal{M}_0 := \left\{ g \in \mathcal{M} \mid b_R = \begin{pmatrix} x \mathbf{1}_n & * \\ 0 & x^{-1} \mathbf{1}_n \end{pmatrix}, b_L = \begin{pmatrix} y^{-1} \sigma & * \\ 0 & y \mathbf{1}_n \end{pmatrix} \right\}, \quad (2.32)$$

where σ is an element of B_n , defined in relation to the previously chosen vector \hat{v} in (2.10) by the property that

$$\sigma \sigma^\dagger = \alpha^2 \mathbf{1}_n + \hat{v} \hat{v}^\dagger. \quad (2.33)$$

This presupposes the condition on the fixed vector \hat{v} that $|\hat{v}|^2 = \alpha^2(\alpha^{-2n} - 1)$, thus ensuring that $\det(\sigma) = 1$. The right-hand part of the corresponding isotropy group is the whole of K_+ . The left-hand part of the isotropy group, denoted $K_+(\sigma)$ (since it depends only on the choice of the element σ), is the direct product

$$K_+(\sigma) = K_+(\hat{w}) \times \mathbb{T}_1, \quad (2.34)$$

with $K_+(\hat{w})$ in (2.14) and with \mathbb{T}_1 given by

$$\mathbb{T}_1 := \{ \hat{\gamma} := \text{diag}(\gamma \mathbf{1}_n, \gamma^{-1} \mathbf{1}_n) \mid \gamma \in \text{U}(1) \}. \quad (2.35)$$

Here, the \mathbb{T}_1 factor of $K_+(\sigma)$ acts on the vector w (2.11) according to the rule

$$\hat{\gamma} : w \mapsto \gamma^{-1} w. \quad (2.36)$$

The task is to characterize the quotient,

$$\mathcal{M}_{\text{red}} := \mathcal{M}_0 / (K_+(\sigma) \times K_+). \quad (2.37)$$

The approach followed in [19] mimics that of [18, 23], and results in a model of \mathcal{M}_{red} (proved in [19] to be a smooth manifold) for which the functions F_l^{red} are presented as a collection of interesting commuting Hamiltonians, and the Φ_l^{red} are trivial. It proceeds, after imposing the constraints, by using the isotropy subgroups for both the left-handed and right-handed actions to bring k to the form

$$k = \begin{pmatrix} \varrho & 0 \\ 0 & \mathbf{1}_n \end{pmatrix} \begin{pmatrix} \cos(q) & i \sin(q) \\ i \sin(q) & \cos(q) \end{pmatrix} \quad \text{with} \quad q = \text{diag}(q_1, \dots, q_n), \quad \varrho \in \text{SU}(n). \quad (2.38)$$

In essence, the result develops from finding the explicit dependence of the matrix Ω as a function of L , i.e. of q , and of conjugate variables, such that the constraint is satisfied.

Alternatively, in the current article we shall look for a model of the reduced phase space for which the functions Φ_l^{red} form a set of interesting commuting Hamiltonians and the F_l^{red} are trivial. This is achieved by using the right-hand isotropy subgroup to bring Ω to blockwise diagonal form, following which the reduction proceeds by representation, via constraints, of the matrix L as a function of Ω and of canonically conjugate variables. Our objective in the next section is to elaborate this statement in detail.

3 Analysis of the reduced system

We start with the observation that, for any $g = kb$ from the constraint surface \mathcal{M}_0 , the right-handed action of K_+ may be used to bring b to the form

$$b = \begin{pmatrix} x\mathbf{1}_n & \beta \\ 0 & x^{-1}\mathbf{1}_n \end{pmatrix} \quad \text{with } \beta = \text{diag}(\beta_1, \dots, \beta_n), \quad \beta_i \in \mathbb{R}, \quad \beta_1 \geq \dots \geq \beta_n \geq 0. \quad (3.1)$$

This is an application of the standard singular value decomposition of $n \times n$ complex matrices. The β_i are invariants on \mathcal{M}_0 with respect to the full gauge group $K_+(\sigma) \times K_+$. Now the idea is to introduce a partial gauge fixing where b has the above form, and label the points of \mathcal{M}_{red} (2.37) by the β_i together with further invariants with respect to the residual gauge transformations. In what follows we *assume* that

$$\beta_1 > \beta_2 > \dots > \beta_n > 0. \quad (3.2)$$

Then the residual gauge group is $K_+(\sigma) \times \mathbb{T}_{n-1}$, where \mathbb{T}_{n-1} contains the matrices of the form $\text{diag}(\tau, \tau)$, with $\tau = \text{diag}(\tau_1, \dots, \tau_n)$ and $\tau_k \in \text{U}(1)$ subject to the condition $\prod_{k=1}^n \tau_k^2 = 1$. It is readily seen that, with $w = w(g)$ defined in (2.11), the triple (β, w, L) provides a complete set of invariants with respect to the factor $K_+(\hat{w})$ of the residual gauge group. After factoring this out, we combine the residual right-handed gauge group \mathbb{T}_{n-1} and the factor \mathbb{T}_1 (2.35) of $K_+(\sigma)$, which acts by (2.36), into the n -torus

$$\mathbb{T}_n = \{T = \text{diag}(\tau, \tau) \mid \tau = \text{diag}(\tau_1, \dots, \tau_n), \quad \tau_i \in \text{U}(1)\}. \quad (3.3)$$

The residual gauge transformation by $T \in \mathbb{T}_n$ acts on the triple (β, w, L) according to

$$T : (\beta, w, L) \mapsto (\beta, Tw, TLT^\dagger). \quad (3.4)$$

In the next subsection, we solve the constraint condition and express w and L , up to the gauge action (3.4), in terms of β and further invariants. In Subsections 2.2 and 2.3 we construct Darboux coordinates on \mathcal{M}_{red} and determine the form of the reduced Hamiltonian Φ_1^{red} in terms of them.

The assumption (3.2) can certainly be made by restriction to an open subset of \mathcal{M}_0 . We shall adopt further similar assumptions in our arguments below; requiring various functions to be non-vanishing before we divide by them. As will be explained in [29], it can be proved that our analysis covers a *dense* open subset of \mathcal{M}_{red} . The domain on which our subsequently derived local formulae are valid is revisited in Section 4.

3.1 Solving the constraint conditions

So far we have introduced partial gauge fixing so that $b = b_R$ takes the form specified in (3.1), and then adopted (3.2). Now we deal with the consequences of the left-hand part of the constraints imposed in (2.32). According to Proposition 2.2, this is equivalent to

$$gg^\dagger - gg^\dagger \begin{pmatrix} 0 & 0 \\ 0 & y^{-2} \end{pmatrix} gg^\dagger = \begin{pmatrix} y^{-2}\sigma\sigma^\dagger & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.5)$$

Substituting $g = kb$, then conjugating with k^\dagger and multiplying by $2y^2$, we have

$$2y^2bb^\dagger - bb^\dagger k^\dagger(id - I)kbb^\dagger = 2k^\dagger \begin{pmatrix} \sigma\sigma^\dagger & 0 \\ 0 & 0 \end{pmatrix} k \quad (3.6)$$

and, after using (2.33) and rewriting the matrix on the right hand side accordingly, we obtain

$$2y^2\Omega - \Omega^2 + \Omega LI\Omega = \alpha^2 id + \alpha^2 LI + 2ww^\dagger. \quad (3.7)$$

Our objective is to find the general solution of (3.7) for L in terms of

$$\Omega = bb^\dagger = \begin{pmatrix} x^2\mathbf{1}_n + \beta^2 & x^{-1}\beta \\ x^{-1}\beta & x^{-2}\mathbf{1}_n \end{pmatrix}. \quad (3.8)$$

Somewhat surprisingly, containing as it does the several unknown quantities, ww^\dagger and L , equation (3.7) can be solved directly. To see this, we start by noticing that the simple blockwise diagonal form of Ω allows us to diagonalise it very easily. To present Ω in diagonalised form, let us introduce the matrix

$$\rho := \begin{pmatrix} \Gamma & \Sigma \\ \Sigma & -\Gamma \end{pmatrix} \quad \text{with} \quad \Gamma := \text{diag}(\Gamma_1, \dots, \Gamma_n), \quad \Sigma := \text{diag}(\Sigma_1, \dots, \Sigma_n). \quad (3.9)$$

Define Γ_i and Σ_i by the formulae

$$\Gamma_i = \left[\frac{\Lambda_i - x^{-2}}{\Lambda_i - \Lambda_i^{-1}} \right]^{\frac{1}{2}}, \quad \Sigma_i = \left[\frac{x^{-2} - \Lambda_i^{-1}}{\Lambda_i - \Lambda_i^{-1}} \right]^{\frac{1}{2}} \quad (3.10)$$

in terms of the new variables

$$\Lambda_1 > \Lambda_2 > \dots > \Lambda_n > \max(x^2, x^{-2}). \quad (3.11)$$

Then it is readily checked that every matrix Ω (3.8) can be written in form

$$\Omega = \rho \text{diag}(\Lambda_1, \dots, \Lambda_n, \Lambda_{n+1}, \dots, \Lambda_{2n}) \rho \quad \text{with} \quad \Lambda_{n+i} = \Lambda_i^{-1}, \quad (3.12)$$

using the following invertible correspondence between the variables β_i and Λ_i :

$$\beta_i = [\Lambda_i + \Lambda_i^{-1} - x^2 - x^{-2}]^{\frac{1}{2}}. \quad (3.13)$$

Because of the blockwise diagonal structure of Ω , it is enough to check the claim for the case $n = 1$. The condition (3.11) is equivalent to (3.2). The relations $\Gamma_i^2 + \Sigma_i^2 = 1$ entail that ρ is a symmetric real orthogonal matrix,

$$\rho = \bar{\rho} = \rho^\dagger = \rho^{-1}. \quad (3.14)$$

Now we return to (3.7), from now on using the variables Λ_i instead of the variables β_i . Setting

$$Q := \rho LI\rho \quad \text{and} \quad \tilde{w} := \rho w, \quad (3.15)$$

we get

$$2y^2\Lambda - \Lambda^2 + \Lambda Q\Lambda = \alpha^2 id + \alpha^2 Q + 2\tilde{w}\tilde{w}^\dagger. \quad (3.16)$$

Assuming that we can divide, this gives in components

$$Q_{ab} = (\Lambda_a\Lambda_b - \alpha^2)^{-1} \left[(\Lambda_a^2 - 2y^2\Lambda_a + \alpha^2)\delta_{ab} + 2\tilde{w}_a\tilde{w}_b^* \right], \quad a, b = 1, 2, \dots, 2n. \quad (3.17)$$

Reformulating (2.15), we have

$$Q\tilde{w} = \tilde{w}; \quad (3.18)$$

that is

$$\tilde{w}_a = (Q\tilde{w})_a = \sum_{b=1}^{2n} Q_{ab}\tilde{w}_b = \frac{(\Lambda_a^2 + \alpha^2 - 2y^2\Lambda_a)}{(\Lambda_a^2 - \alpha^2)}\tilde{w}_a + 2\tilde{w}_a \sum_{b=1}^{2n} \frac{|\tilde{w}_b|^2}{\Lambda_a\Lambda_b - \alpha^2}. \quad (3.19)$$

Supposing that $\tilde{w}_a \neq 0$, this yields

$$\sum_{b=1}^{2n} \frac{|\tilde{w}_b|^2}{\Lambda_a\Lambda_b - \alpha^2} = \frac{y^2\Lambda_a - \alpha^2}{\Lambda_a^2 - \alpha^2}, \quad (3.20)$$

from which each of the $|\tilde{w}_b|^2$ is expressed in terms of the components of Λ , by means of the inverse of the Cauchy-like matrix $C_{ab} = (\Lambda_a\Lambda_b - \alpha^2)^{-1}$.

Working on the open domain where (3.2) and all non-vanishing assumptions hold, we find explicit expressions for $|\tilde{w}_a|^2$ as functions of Λ .

Proposition 3.1. *Solving (3.20), we obtain*

$$|\tilde{w}_a|^2 = \alpha(\Lambda_a - y^2) \prod_{\substack{b=1 \\ (b \neq a)}}^{2n} \frac{\alpha^{-1}\Lambda_a\Lambda_b - \alpha}{\Lambda_a - \Lambda_b}, \quad a = 1, \dots, 2n. \quad (3.21)$$

Proof. Rewriting (3.20), we have

$$|\tilde{w}_a|^2 = \sum_{b=1}^{2n} (C^{-1})_{ab} \frac{\alpha^{-1}y^2x_b - 1}{x_b^2 - 1}, \quad (3.22)$$

with

$$C_{ab} = \frac{\alpha^{-2}}{x_ax_b - 1}, \quad x_a = \alpha^{-1}\Lambda_a. \quad (3.23)$$

From the standard formula for the inverse of a Cauchy matrix, we may deduce

$$(C^{-1})_{ab} = \alpha^2 \frac{(x_ax_b)^{2n}}{(x_ax_b - 1)} \frac{A(x_a^{-1})A(x_b^{-1})}{A'(x_a)A'(x_b)}, \quad a, b = 1, 2, \dots, 2n, \quad (3.24)$$

using the complex function

$$A(z) = \prod_{a=1}^{2n} (z - x_a) \quad (3.25)$$

and its derivative $A'(z)$. Consequently,

$$|\tilde{w}_a|^2 = \frac{\alpha^2 x_a^{2n} A(x_a^{-1})}{A'(x_a)} \sum_{b=1}^{2n} \frac{x_b^{2n} A(x_b^{-1})}{(x_ax_b - 1)A'(x_b)} \frac{\alpha^{-1}y^2x_b - 1}{x_b^2 - 1}. \quad (3.26)$$

To simplify the sum, introduce the rational function $\Psi_a(z)$ of a complex variable

$$\Psi_a(z) := \frac{z^{2n} A(z^{-1})(\alpha^{-1}y^2z - 1)}{(x_az - 1)(z^2 - 1)A(z)}. \quad (3.27)$$

Observing that $\Psi_a(z)dz$ extends to a meromorphic 1-form on the Riemann sphere $\overline{\mathbb{C}}$, the sum of its residues over $\overline{\mathbb{C}}$ must be zero. All the poles of $\Psi_a(z)dz$ are simple, and they are located

at $z = x_b$ for $b = 1, 2, \dots, 2n$ and at $z = \pm 1$. The sum of the residues at $z = x_b$ is exactly the sum in (3.26), and so this sum can be evaluated by computing the residues at $z = \pm 1$. We find

$$\operatorname{Res}_{z=+1}(\Psi_a(z)dz) + \operatorname{Res}_{z=-1}(\Psi_a(z)dz) = -\frac{x_a - \alpha^{-1}y^2}{x_a^2 - 1}. \quad (3.28)$$

Substitution into (3.26) produces

$$|\tilde{w}_a|^2 = \alpha \left(\frac{\alpha x_a - y^2}{x_a^2 - 1} \right) \frac{x_a^{2n} A(x_a^{-1})}{A'(x_a)}, \quad (3.29)$$

and replacing $x_a = \alpha^{-1}\Lambda_a$ gives the stated result. \square

We have expressed Q (3.15), and therefore also $L = \rho Q \rho I$, in terms of Λ and $\tilde{w} = \rho w$. Hence it follows from (3.13) and the transformation rule (3.4) that we may parametrize the gauge orbits using Λ together with invariants of w . Equivalently, we may build invariants out of \tilde{w} , which, due to the form of ρ (3.9), transforms under the residual gauge action (3.4) in the same way as w , i.e.,

$$T : \tilde{w} \mapsto T\tilde{w}. \quad (3.30)$$

Recalling the form of $T \in \mathbb{T}_n$ (3.3), we see that the angles θ_j defined by the relations

$$\tilde{w}_j^* \tilde{w}_{n+j} = |\tilde{w}_j \tilde{w}_{n+j}| e^{i\theta_j}, \quad j = 1, \dots, n, \quad (3.31)$$

are invariants. Since the conditions $\tilde{w}_j \in \mathbb{R}_{>0}$ for all $j = 1, \dots, n$ define a complete gauge fixing for the residual gauge transformations (3.4), the variables Λ_j together with the θ_j provide a *complete set of invariants* that label the gauge orbits in our open subset of \mathcal{M}_0 .

3.2 Darboux coordinates on the reduced space

The reduced phase space \mathcal{M}_{red} is a symplectic manifold, and we denote the Poisson bracket of smooth functions on \mathcal{M}_{red} by $\{ , \}_{\text{red}}$. It is apparent already in (2.26) that the eigenvalues of Ω and the phase-like invariants of \tilde{w} , as exhibited in (3.31), are candidates for Darboux coordinates. We are going to prove that they indeed are such. As a preparation, we next formulate a consequence of the general theory of Hamiltonian reduction.

Let \mathcal{M}_1 denote the subspace of the constraint surface \mathcal{M}_0 (2.32) consisting of the elements for which b has the form (3.1). Then there is a natural one-to-one correspondence between the gauge invariant smooth functions on \mathcal{M}_1 , with respect to the residual gauge transformations acting on \mathcal{M}_1 , and the smooth functions on \mathcal{M}_{red} (2.37). Take a $(K_+ \times K_+)$ -invariant function \mathcal{H} on \mathcal{M} and a gauge invariant function \mathcal{G} on \mathcal{M}_1 , and consider the Poisson bracket $\{\mathcal{G}^{\text{red}}, \mathcal{H}^{\text{red}}\}_{\text{red}}$ of the corresponding functions \mathcal{G}^{red} and \mathcal{H}^{red} on \mathcal{M}_{red} . The gauge invariant function on \mathcal{M}_1 that corresponds to $\{\mathcal{G}^{\text{red}}, \mathcal{H}^{\text{red}}\}_{\text{red}}$ is the derivative of \mathcal{G} along any vector field of the form

$$\mathbb{X}_{\mathcal{H}}^1 = \mathbb{X}_{\mathcal{H}} + \mathbb{Y}_{\mathcal{H}}, \quad (3.32)$$

where $\mathbb{X}_{\mathcal{H}}$ is the Hamiltonian vector field of \mathcal{H} restricted to \mathcal{M}_1 , and $\mathbb{Y}_{\mathcal{H}}$ represents the right-handed action of point dependent elements of the Lie algebra \mathfrak{k}_+ of K_+ , chosen in such a way that $\mathbb{X}_{\mathcal{H}}^1$ is tangent to \mathcal{M}_1 . This is expressed by the equality

$$\{\mathcal{G}^{\text{red}}, \mathcal{H}^{\text{red}}\}_{\text{red}} = (\mathbb{X}_{\mathcal{H}}^1(\mathcal{G}))^{\text{red}}. \quad (3.33)$$

The vector field $\mathbb{X}_{\mathcal{H}}^1$ is determined in the following way. If \dot{k} and \dot{b} denote the components of $\mathbb{X}_{\mathcal{H}}(g)$ corresponding to the decomposition $\mathcal{M} \ni g = kb$, and k' and b' denote the components of $\mathbb{X}_{\mathcal{H}}^1(g)$ corresponding to the decomposition $\mathcal{M}_1 \ni g = kb$, then we have

$$k' = \dot{k} - kY, \quad b' = \dot{b} + [Y, b], \quad (3.34)$$

where $Y \in \mathfrak{k}_+$ is the ‘‘compensating infinitesimal gauge transformation’’, ensuring that the $\mathbb{X}_{\mathcal{H}}^1$ -derivative b' of b is consistent with the form of b (3.1). This fixes Y up to infinitesimal, right-handed gauge transformations tangent to \mathcal{M}_1 . Concretely, writing $Y = \text{diag}(Y_1, Y_2)$, the B -component of (3.34) can be recast as

$$\beta' = \dot{b}_{12} + Y_1\beta - \beta Y_2, \quad (3.35)$$

where \dot{b}_{12} denotes the top-right $n \times n$ block of \dot{b} . The condition on Y is that β' must be a real diagonal matrix, because β is a real diagonal matrix. We observe from (3.35) that, up to its inherent ambiguity, Y can be viewed as a function of β and \dot{b}_{12} , which themselves are functions on \mathcal{M}_1 .

We shall apply the above procedure to the open submanifold $\check{\mathcal{M}}_{\text{red}}$ of \mathcal{M}_{red} that can be parametrized by the invariants Λ_j (3.12) and $e^{i\theta_j}$ (3.31), and denote the corresponding submanifold of \mathcal{M}_1 by $\check{\mathcal{M}}_1$. We note that every gauge invariant function on $\check{\mathcal{M}}_1$ can be regarded as a function of β and w , since they determine L by equations (3.13)-(3.17). For a gauge invariant function \mathcal{G} on $\check{\mathcal{M}}_1$, denoting by \mathcal{G}^{red} the expression in the local coordinates $(\Lambda, e^{i\theta})$ of the corresponding function on $\check{\mathcal{M}}_{\text{red}}$, we have

$$\mathcal{G}^{\text{red}}(\Lambda, e^{i\theta}) = \mathcal{G}(\beta, w), \quad (3.36)$$

where $(\beta, w) \mapsto (\Lambda, e^{i\theta})$ is given by (3.13), (3.15) and (3.31). We shall also use the fact that on $\check{\mathcal{M}}_1$ the functions $|\tilde{w}_a|$ ($a = 1, \dots, 2n$) are non-zero and depend only on Λ .

Theorem 3.2. *On the open submanifold of $\check{\mathcal{M}}_{\text{red}} \subset \mathcal{M}_{\text{red}}$ parametrized by $\lambda_j := \frac{1}{2} \log \Lambda_j$ (3.12) and the angles θ_j (3.31) we have the canonical Poisson brackets*

$$\{\lambda_j, \lambda_l\}_{\text{red}} = 0, \quad \{\theta_j, \lambda_l\}_{\text{red}} = \delta_{jl}, \quad \{\theta_j, \theta_l\}_{\text{red}} = 0, \quad j, l = 1, \dots, n. \quad (3.37)$$

Proof. The first two relations in (3.37) are shown easily. For this, we start by pointing out that the reductions of the Poisson commuting functions $F_l \in C^\infty(\mathcal{M})^{K_+ \times K_+}$, defined in (2.21), read

$$F_l^{\text{red}} = \frac{1}{l} \sum_{j=1}^n \cosh(2l\lambda_j). \quad (3.38)$$

The identity $\{F_j^{\text{red}}, F_l^{\text{red}}\}_{\text{red}} = 0$ for all j, l is assured by the reduction, and is clearly equivalent to $\{\lambda_j, \lambda_l\}_{\text{red}} = 0$.

Direct calculation on the reduced phase space gives

$$\{e^{i\theta_j}, F_l^{\text{red}}\}_{\text{red}} = 2ie^{i\theta_j} \sum_{m=1}^n \{\theta_j, \lambda_m\}_{\text{red}} \sinh(2l\lambda_m). \quad (3.39)$$

Notice from (2.24) that the Hamiltonian vector field of F_l is tangent to \mathcal{M}_1 . Calculating the right-hand-side of (3.33) for $\mathcal{H} = F_l$ and $\mathcal{G} = e^{i\theta_j}$ defined by (3.31), we find from (2.26) that

$$\mathbb{X}_{F_l}^1(e^{i\theta_j}) = 2ie^{i\theta_j} \sinh(2l\lambda_j). \quad (3.40)$$

Equality between the last two expressions is equivalent to $\{\theta_j, \lambda_l\}_{\text{red}} = \delta_{jl}$.

The Jacobi identity for $\{, \}_{\text{red}}$ and the formulae $\{\theta_i, \lambda_j\}_{\text{red}} = \delta_{ij}$ imply that the functions

$$P_{kl} := \{\theta_k, \theta_l\}_{\text{red}} \quad (3.41)$$

depend only on λ . It remains to prove that these functions vanish identically.

We consider the function $\Phi_1 \in C^\infty(\mathcal{M})^{K_+ \times K_+}$, also defined in (2.21). The Hamiltonian vector field of Φ_1 , given by the $l = 1$ special case of (2.25), is tangent to \mathcal{M}_0 , but is not tangent to $\check{\mathcal{M}}_1$. In this case $\dot{b}_{12} = 2ix^{-1}L_{12}$, and we can find $Y = Y(\beta, 2ix^{-1}L_{12}) \in \mathfrak{k}_+$ such that

$$\beta' \equiv \mathbb{X}_{\Phi_1}^1(\beta) = 2ix^{-1}L_{12} + Y_1\beta - \beta Y_2 \quad (3.42)$$

will be a real diagonal matrix. To proceed further, we point out that for every element $g = kb \in \check{\mathcal{M}}_1$, there exists another element $g^\sharp = k^\sharp b \in \check{\mathcal{M}}_1$ for which

$$w(g^\sharp) = w(g)^* \quad \text{and consequently} \quad L(g^\sharp) = L(g)^*, \quad (3.43)$$

where star denotes complex conjugation. This holds since the constraint condition (3.7) is stable under complex conjugation². More concretely, it reflects the fact that for fixed β the constraints determine only the moduli $|\tilde{w}_a|$ of the \tilde{w}_a (3.15), and all values are possible for $\arg(\tilde{w}_a)$. For a given g , any two choices of g^\sharp are related by a gauge transformation, since w determines k up to the left-handed action of $K_+(\hat{w})$. The rest of the proof relies on the property

$$Y(\beta, 2ix^{-1}L_{12}^*) = Y(\beta, 2ix^{-1}L_{12})^T, \quad (3.44)$$

which follows by comparison of equation (3.42) with its complex conjugate. Of course, this equality is understood up to the ambiguity in Y , that does not affect the derivatives of gauge invariant functions.

Let $A = \text{diag}(A_1, A_2, \dots, A_n)$ be a diagonal matrix with $A_j \in \mathbb{R}$ for all j , and introduce the $2n \times 2n$ matrix

$$\hat{A} = \begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix}. \quad (3.45)$$

We then define the gauge invariant function \mathcal{G}_A on $\check{\mathcal{M}}_1$ by

$$\mathcal{G}_A(g) = \frac{1}{2i} w^\dagger \hat{A} w. \quad (3.46)$$

Using the $l = 1$ case of \dot{w} from (2.27), with (3.34) and (2.15), the derivative w' of w along $\mathbb{X}_{\Phi_1}^1$ reads

$$w' = \frac{1}{2}i(id + I)L(id - I)w + Yw, \quad (3.47)$$

and we easily check that

$$\mathbb{X}_{\Phi_1}^1(\mathcal{G}_A)(g) = \mathbb{X}_{\Phi_1}^1(\mathcal{G}_A)(g^\sharp). \quad (3.48)$$

Indeed, denoting $Y(\beta, 2ix^{-1}L_{12})$ simply by Y for short, we have

$$\begin{aligned} \mathbb{X}_{\Phi_1}^1(\mathcal{G}_A)(g) &= w^\dagger \hat{A} w' + w'^\dagger \hat{A} w \\ &= \frac{1}{2i} w^\dagger \left(\left[\frac{1}{2}i(id - I)L^\dagger(id + I) + Y \right] \hat{A}^T + \hat{A} \left[\frac{1}{2}i(id + I)L(id - I) + Y \right] \right) w \end{aligned} \quad (3.49)$$

²We can take $g^\sharp = g^*$ whenever the fixed vector \hat{w} (2.10) is real.

and, using (3.43) and (3.44),

$$\mathbb{X}_{\Phi_1}^1(\mathcal{G}_A)(g^\sharp) = \frac{1}{2i} w^T \left(\hat{A} \left[\frac{1}{2} i(id + I)L^*(id - I) + Y^T \right] + \left[\frac{1}{2} i(id - I)L^T(id + I) + Y^T \right] \hat{A}^T \right) w^*. \quad (3.50)$$

It is easy to see that these are the same.

Next, let us inspect the reduced version of the equality (3.48). Taking into account the relation $\tilde{w} = \rho w$ and using $\rho \hat{A} \rho = -\hat{A}$, we obtain

$$\mathcal{G}_A^{\text{red}}(\lambda, \theta) = \sum_{i=1}^n A_i (|\tilde{w}_i| |\tilde{w}_{n+i}|)(\lambda) \sin \theta_i. \quad (3.51)$$

On the other hand, Φ_1^{red} takes the form

$$\Phi_1^{\text{red}}(\lambda, \theta) = V(\lambda) + \sum_{j=1}^n f_j(\lambda) \cos \theta_j \quad (3.52)$$

with some functions V and f_j . (Equation (3.59) below shows that $f_j(\lambda) \neq 0$ on $\check{\mathcal{M}}_{\text{red}}$.) Direct calculation then yields

$$\begin{aligned} \{\mathcal{G}_A^{\text{red}}, \Phi_1^{\text{red}}\}_{\text{red}} &= \sum_{i=1}^n \left[\frac{\partial \mathcal{G}_A^{\text{red}}}{\partial \theta_i} \frac{\partial \Phi_1^{\text{red}}}{\partial \lambda_i} - \frac{\partial \mathcal{G}_A^{\text{red}}}{\partial \lambda_i} \frac{\partial \Phi_1^{\text{red}}}{\partial \theta_i} \right] + \sum_{i,j=1}^n P_{ij} \frac{\partial \mathcal{G}_A^{\text{red}}}{\partial \theta_i} \frac{\partial \Phi_1^{\text{red}}}{\partial \theta_j} \\ &= \sum_{i=1}^n \sum_{j=1}^n A_j f_i \frac{\partial (|\tilde{w}_j| |\tilde{w}_{j+1}|)}{\partial \lambda_i} \sin \theta_i \sin \theta_j \\ &\quad + \sum_{i=1}^n A_i |\tilde{w}_i| |\tilde{w}_{n+i}| \cos \theta_i \left[\sum_{j=1}^n \frac{\partial f_j}{\partial \lambda_i} \cos \theta_j + \frac{\partial V}{\partial \lambda_i} - \sum_{j=1}^n f_j P_{ij} \sin \theta_j \right], \end{aligned} \quad (3.53)$$

with the notation (3.41). This implies the relation

$$\{\mathcal{G}_A^{\text{red}}, \Phi_1^{\text{red}}\}_{\text{red}}(\lambda, -\theta) - \{\mathcal{G}_A^{\text{red}}, \Phi_1^{\text{red}}\}_{\text{red}}(\lambda, \theta) = 2 \sum_{i=1}^n \sum_{j=1}^n A_i \cos \theta_i \left(|\tilde{w}_i| |\tilde{w}_{n+i}| P_{ij} f_j \right)(\lambda) \sin \theta_j. \quad (3.54)$$

Now we notice from (3.31) that, for invariant functions on $\check{\mathcal{M}}_1$, $(\lambda, \theta) \mapsto (\lambda, -\theta)$ is equivalent to $\tilde{w} \mapsto \tilde{w}^*$ and, as $w = \rho \tilde{w}$, the same is true for w , i.e. $w \mapsto w^*$. Therefore, taking into account also (3.33) and (3.43), the reduced version of the equality (3.48) says that the expression in (3.54) is zero. Choosing

$$A_i = \delta_{ik}, \quad \theta_j = -\frac{\pi}{2} \delta_{jl} \quad (3.55)$$

we obtain

$$2|\tilde{w}_k| |\tilde{w}_{n+k}| f_l P_{kl} = 0. \quad (3.56)$$

This necessitates the vanishing of P_{kl} , whence the proof is complete. \square

3.3 The form of the Hamiltonian Φ_1^{red}

The Hamiltonian of interest is the reduction of Φ_1 —the simplest element in the ring of invariant functions of L —expressed as a function of the Darboux coordinates λ_j, θ_j (3.37) on the reduced phase space. The desired expression can be derived by evaluation of the formula

$$\Phi_1^{\text{red}}(\lambda, \theta) \simeq \frac{1}{2} \text{tr } L|_{\check{\mathcal{M}}_1} \quad (3.57)$$

using, on account of (3.15), $L = \rho Q \rho I$ with Q given by (3.17). Since $\text{tr } L$ is gauge invariant, we obtain Φ_1^{red} as a function of λ, θ if we substitute (3.21) and (3.31). In agreement with [19], let us replace

$$\alpha = e^{-\mu}, \quad x = e^{-v}, \quad y = e^{-u}, \quad (3.58)$$

where u, v, μ are real parameters, $\mu > 0$. We shall prove the following

Theorem 3.3. *The reduced Hamiltonian Φ_1^{red} takes the form*

$$\begin{aligned} \Phi_1^{\text{red}}(\lambda, \theta) = & V(\lambda) + e^{v-u} \sum_{k=1}^n \frac{\cos \theta_k}{\cosh^2 \lambda_k} \left[1 - \frac{\sinh^2 v}{\sinh^2 \lambda_k} \right]^{1/2} \left[1 - \frac{\sinh^2 u}{\sinh^2 \lambda_k} \right]^{1/2} \\ & \times \prod_{\substack{l=1 \\ (l \neq k)}}^n \left[1 - \frac{\sinh^2 \mu}{\sinh^2(\lambda_k - \lambda_l)} \right]^{1/2} \left[1 - \frac{\sinh^2 \mu}{\sinh^2(\lambda_k + \lambda_l)} \right]^{1/2} \end{aligned} \quad (3.59)$$

with

$$V(\lambda) = e^{v-u} \left(\frac{\sinh(v) \sinh(u)}{\sinh^2 \mu} \prod_{k=1}^n \left[1 - \frac{\sinh^2 \mu}{\sinh^2 \lambda_k} \right] - \frac{\cosh(v) \cosh(u)}{\sinh^2 \mu} \prod_{k=1}^n \left[1 + \frac{\sinh^2 \mu}{\cosh^2 \lambda_k} \right] + C \right) \quad (3.60)$$

where $C = ne^{u-v} + \frac{\cosh(v-u)}{\sinh^2 \mu}$.

Proof. Let us write

$$Q = D + 2WCW^\dagger \quad (3.61)$$

where, from (3.17),

$$\begin{aligned} D_{ab} = \delta_{ab} D_a \quad \text{with} \quad D_a = (\Lambda_a^2 - \alpha^2)^{-1} (\Lambda_a^2 + \alpha^2 - 2y^2 \Lambda_a), \\ \mathcal{W}_{ab} = \tilde{w}_a \delta_{ab}, \quad \text{and} \quad C_{ab} = (\Lambda_a \Lambda_b - \alpha^2)^{-1}. \end{aligned} \quad (3.62)$$

Hence, using (3.15) together with (3.9), we have

$$\begin{aligned} \Phi_1^{\text{red}} &= \frac{1}{2} \text{tr } Q \rho I \rho = \frac{1}{2} \text{tr} (D + 2WCW^\dagger) \begin{pmatrix} \Gamma^2 - \Sigma^2 & 2\Gamma\Sigma \\ 2\Gamma\Sigma & -\Gamma^2 + \Sigma^2 \end{pmatrix} \\ &= \frac{1}{2} \sum_{k=1}^n (\Gamma_k^2 - \Sigma_k^2) \left[D_k - D_{n+k} + 2C_{kk} |\tilde{w}_k|^2 - 2C_{n+k, n+k} |\tilde{w}_{n+k}|^2 \right] \\ &\quad + 2 \sum_{k=1}^n \Gamma_k \Sigma_k C_{k, n+k} (\tilde{w}_k \tilde{w}_{n+k}^* + \tilde{w}_k^* \tilde{w}_{n+k}). \end{aligned} \quad (3.63)$$

Substituting from (3.62), (3.10) and then reorganising terms, we get

$$\begin{aligned} \Phi_1^{\text{red}} &= \frac{1}{2} \sum_{a=1}^{2n} \frac{\Lambda_a + \Lambda_a^{-1} - 2x^{-2}}{\Lambda_a - \Lambda_a^{-1}} \left(\frac{\Lambda_a^2 + \alpha^2 - 2y^2 \Lambda_a}{\Lambda_a^2 - \alpha^2} + \frac{2|\tilde{w}_a|^2}{\Lambda_a^2 - \alpha^2} \right) \\ &\quad + 4 \sum_{k=1}^n \left[\frac{(\Lambda_k - x^{-2})(x^{-2} - \Lambda_k^{-1})}{(\Lambda_k - \Lambda_k^{-1})^2} \right]^{\frac{1}{2}} \frac{|\tilde{w}_k| |\tilde{w}_{n+k}| \cos \theta_k}{1 - \alpha^2}. \end{aligned} \quad (3.64)$$

Let us denote by V the first sum in formula (3.64), and insert $|\tilde{w}_a|^2$ from (3.21). Introducing the complex function $\Psi(z)$ by

$$\Psi(z) = F(z) + G(z) \quad (3.65)$$

with

$$\begin{aligned} F(z) &= \alpha^2 \frac{(z^2 - 2x^{-2}z + 1)(z - y^2)}{(z^2 - 1)(z^2 - \alpha^2)^2} \prod_{a=1}^{2n} \frac{(\alpha^{-1}z\Lambda_a - \alpha)}{(z - \Lambda_a)}, \\ G(z) &= \frac{1}{2} \frac{(z^2 - 2x^{-2}z + 1)(z^2 + \alpha^2 - 2y^2z)}{(z^2 - 1)(z^2 - \alpha^2)} \sum_{a=1}^{2n} \frac{1}{z - \Lambda_a}, \end{aligned} \quad (3.66)$$

observe that

$$V = \sum_{a=1}^{2n} \operatorname{Res}_{z=\Lambda_a} \left(\Psi(z) dz \right). \quad (3.67)$$

As $\Psi(z)dz$ extends to a meromorphic 1-form on the Riemann sphere $\overline{\mathbb{C}}$, the sum of its residues over the poles in $\overline{\mathbb{C}}$ is zero. In addition to $z = \Lambda_a$ for $a = 1, \dots, 2n$, $\Psi(z)dz$ possesses poles at $z = \pm 1, \pm \alpha, \infty$. All residues can be calculated straightforwardly. In this way, using also the substitutions $\Lambda_j = e^{2\lambda_j}$, (3.58) and elementary hyperbolic identities like $\sinh(\nu + \mu) \sinh(\nu - \mu) = \sinh^2 \nu - \sinh^2 \mu$, we obtain formula (3.60) for V .

To finish the derivation, we first rewrite (3.21) as

$$|\tilde{w}_k|^2 = e^{-\mu} (e^{2\lambda_k} - y^2) \frac{\sinh(\mu)}{\sinh(2\lambda_k)} \prod_{\substack{i=1 \\ (i \neq k)}}^n \left(\frac{\sinh(\lambda_k + \lambda_i + \mu) \sinh(\lambda_k - \lambda_i + \mu)}{\sinh(\lambda_k - \lambda_i) \sinh(\lambda_k + \lambda_i)} \right) \quad (3.68)$$

and

$$|\tilde{w}_{n+k}|^2 = e^{-\mu} (y^2 - e^{-2\lambda_k}) \frac{\sinh(\mu)}{\sinh(2\lambda_k)} \prod_{\substack{i=1 \\ (i \neq k)}}^n \left(\frac{\sinh(\lambda_k + \lambda_i - \mu) \sinh(\lambda_k - \lambda_i - \mu)}{\sinh(\lambda_k - \lambda_i) \sinh(\lambda_k + \lambda_i)} \right) \quad (3.69)$$

for $k = 1, \dots, n$. Substituting these in the second term of (3.64) and using again (3.58) leads to the claimed formula (3.59) for Φ_1^{red} . \square

4 Discussion

The Heisenberg double \mathcal{M} of the Poisson Lie group $K = \text{SU}(2n)$, equipped with the Abelian Poisson algebras generated by $\{F_l\}$ and $\{\Phi_l\}$ (2.21), permits Hamiltonian reduction by the constraint in (2.32). All the functions F_l and Φ_l are invariant with respect to the symmetry group $K_+ \times K_+$, and thus $\{F_l\}$ and $\{\Phi_l\}$ descend to Abelian Poisson algebras on the reduced phase space \mathcal{M}_{red} (2.37), where they engender two Liouville integrable systems. The present paper continues the line of research started in [18] and further advanced in [19, 21, 22]. The aim of these studies is to achieve detailed understanding of the integrable systems defined by the collections of reduced Hamiltonians $\{F_l^{\text{red}}\}$ and $\{\Phi_l^{\text{red}}\}$ as well as their analogues obtained by using $\text{SU}(n, n)$ instead of $\text{SU}(2n)$ in the decompositions (2.1), (2.2). The pertinent reductions admit two natural models for the reduced phase space, which are associated with two systems of Darboux coordinates on (dense open submanifolds of) \mathcal{M}_{red} . The Darboux coordinates emerge from the eigenvalues of two matrices complemented by their respective

canonical conjugates. In our setting these two matrices are Ω and L (2.7). The coordinates based on diagonalization of L were described in [19], following [18]. Here, we have constructed alternative Darboux coordinates utilizing the eigenvalues $\Lambda_j = e^{2\lambda_j}$ of Ω .

The canonical conjugates of the variables λ_j are angles θ_j , parametrizing an n -torus \mathbb{T}^n , but so far we have not specified the range of the eigenvalue-parameters λ_j : it will be proved in [29] that their full range is the closure of the convex polyhedron

$$\mathcal{D}_+^\lambda = \{\lambda \in \mathbb{R}^n \mid \lambda_1 > \lambda_2 > \cdots > \lambda_n > \max(|v|, |u|), \lambda_i - \lambda_{i+1} > \mu, i = 1, \dots, n-1\}, \quad (4.1)$$

where μ , u and v are the constants (3.58) appearing in the definition of the constraint (2.32). The restriction of λ to the domain \mathcal{D}_+^λ is a consequence of the facts that the variables $\Lambda_j = e^{2\lambda_j}$ satisfy (3.11) and that the functions $|\tilde{w}_a|^2$ in (3.21) cannot be negative. Indeed, these functions, exhibited also in (3.68)-(3.69), are all positive precisely on the domain (4.1).

We have seen that the reduced Hamiltonian Φ_1^{red} takes the interesting RSvD form (3.59) in terms of the Darboux coordinates attached to $\mathcal{D}_+^\lambda \times \mathbb{T}^n = \{(\lambda, e^{i\theta})\}$. On the other hand, in these coordinates the reduced Hamiltonians F_l^{red} depend only on λ , as given by (3.38). This means that λ_j, θ_j are action-angle variables for the Liouville integrable system $\{F_l^{\text{red}}\}$, and the θ -tori are just the Liouville tori. The boundary of the polyhedron \mathcal{D}_+^λ actually corresponds to lower-dimensional Liouville tori.

Now we recall the other system of Darboux coordinates, denoted (\hat{p}, \hat{q}) in [19]. The \hat{q}_j are angles, whereas the \hat{p}_j are related to the parameters q_j of the generalized Cartan decomposition of $k \in K$ utilized to obtain the formula (2.38). Concretely [18, 19], we have

$$e^{\hat{p}_j} = \sin(q_j). \quad (4.2)$$

These variables encode the eigenvalues of $L = k^\dagger I k I$ since L is conjugate to the matrix

$$\begin{pmatrix} \cos(2q) & i \sin(2q) \\ i \sin(2q) & \cos(2q) \end{pmatrix}, \quad q = \text{diag}(q_1, \dots, q_n). \quad (4.3)$$

The range of the variables \hat{p}_j can be shown [19]³ to be the closure of the domain

$$\mathcal{D}_+^{\hat{p}} = \{\hat{p} \in \mathbb{R}^n \mid \hat{p}_1 < \min(0, v-u), \hat{p}_j - \hat{p}_{j+1} > \mu (j = 1, \dots, n-1)\}. \quad (4.4)$$

The pair $(\hat{p}, e^{i\hat{q}})$ filling the domain $\mathcal{D}_+^{\hat{p}} \times \mathbb{T}^n$ yields Darboux coordinates on a dense open subset of \mathcal{M}_{red} , and in these coordinates the Hamiltonians Φ_l^{red} become trivial, while F_1^{red} gives an interesting Hamiltonian of RSvD type. Specifically, one obtains

$$\Phi_l^{\text{red}} = \frac{1}{l} \sum_{j=1}^n \cos(2lq_j(\hat{p})), \quad (4.5)$$

referring to (4.2), and

$$F_1^{\text{red}} = U(\hat{p}) - \sum_{j=1}^n \cos(\hat{q}_j) U_1(\hat{p}_j)^{\frac{1}{2}} \prod_{\substack{k=1 \\ (k \neq j)}}^n \left[1 - \frac{\sinh^2(\mu)}{\sinh^2(\hat{p}_j - \hat{p}_k)} \right]^{\frac{1}{2}} \quad (4.6)$$

with

$$U(\hat{p}) = \frac{e^{-2u} + e^{2v}}{2} \sum_{j=1}^n e^{-2\hat{p}_j}, \quad U_1(\hat{p}_j) = [1 - (1 + e^{2(v-u)})e^{-2\hat{p}_j} + e^{2(v-u)}e^{-4\hat{p}_j}]. \quad (4.7)$$

³In this reference the unnecessary assumption $v > u$ was made.

Hence \hat{p}_j, \hat{q}_j are action-angle variables for the Liouville integrable system $\{\Phi_l^{\text{red}}\}$, and the \hat{p}_j serve also as position variables for F_1^{red} (4.6). Incidentally, it is manifest from the identity

$$U_1(\hat{p}_j) = 4e^{v-u} e^{-2\hat{p}_j} \sinh(\hat{p}_j) \sinh(\hat{p}_j + u - v) \quad (4.8)$$

that the Hamiltonian F_1^{red} (4.6) is real on the domain (4.4), as it must be on account of its action-angle form (3.38).

We conclude from the above that the Liouville integrable systems $\{F_l^{\text{red}}\}$ and $\{\Phi_l^{\text{red}}\}$ are *in action-angle duality*. Indeed, F_1^{red} takes the RSvD form (4.6) in terms of the action-angle variables of $\{\Phi_l^{\text{red}}\}$, and Φ_1^{red} is given by the other RSvD type formula (3.59) in terms of the action-angle variables of $\{F_l^{\text{red}}\}$.

As was mentioned in the Introduction, the first systematic investigation of action-angle duality relied on direct methods [13, 14]. Since then, the reduction interpretation of most (although still not all) examples of Ruijsenaars have been found, and also several new cases of action-angle duality were unearthed utilizing this method; see [16, 17, 24, 25] and references therein. The present paper should be seen as a contribution to the research goal to describe dual pairs for all RSvD type systems in reduction terms.

Global properties of the reduced phase space (2.37) and consequences of the duality for the dynamics will be studied in our subsequent publication [29]. The relation of F_1^{red} (4.6) to the five-parameter family of RSvD Hamiltonians [6] was described in [19], and in [29] we will also present such a connection for Φ_1^{red} (3.59). We here only note (see Appendix A) that Φ_1^{red} is a deformation of the action-angle dual of the trigonometric BC_n Sutherland Hamiltonian, as must be the case since F_1^{red} can be viewed as a deformation of the latter [18, 19].

We wish to point out that their reduction origin naturally associates Lax matrices to the models obtained, basically because Ω and L (2.7) generate the commuting Hamiltonians (2.21) before reduction. Recently there appeared new results about Lax matrices for certain hyperbolic RSvD models [30], and it would be interesting to compare those with the Lax matrices that arise in our setting.

We finally remark that the quantum mechanical (bispectral) analogue of our dual pair should be understood. The recent paper by van Diejen and Emsiz [31] is certainly relevant for finding the answer to this question. We hope that our investigations will be developed in several directions in the future, including bispectral aspects withal.

Acknowledgements. This work was supported in part by the Hungarian Scientific Research Fund (OTKA) under the grant K-111697.

A Connection to the dual of the BC_n Sutherland model

In this appendix we present the ‘‘cotangent bundle limit’’ of the Hamiltonian Φ_1^{red} (3.59). We find it convenient to introduce the notation

$$H_1(\lambda, \theta; u, v, \mu) := \Phi_1^{\text{red}}(\lambda, \theta) \quad (\text{A.1})$$

containing the coupling parameters u, v, μ as given in (3.59). Let us now take any positive parameter r and consider the one-parameter family of Hamiltonians

$$H_r(\lambda, \theta; u, v, \mu) := H_1(r\lambda, \theta; ru, rv, r\mu), \quad (\text{A.2})$$

which are defined, for any $r > 0$, on the same domain $\mathcal{D}_+^\lambda \times \mathbb{T}^n$ (4.1) as H_1 . It is easy to check that H_r has a limit as $r \rightarrow 0$. Indeed, we obtain

$$\lim_{r \rightarrow 0} H_r(\lambda, \theta; u, v, \mu) = H_0(\lambda, \theta; u, v, \mu) \quad (\text{A.3})$$

with

$$\begin{aligned} H_0(\lambda, \theta; u, v, \mu) = & V_0(\lambda; u, v, \mu) + \sum_{k=1}^n \cos(\theta_k) \left[1 - \frac{v^2}{\lambda_k^2}\right]^{1/2} \left[1 - \frac{u^2}{\lambda_k^2}\right]^{1/2} \\ & \times \prod_{\substack{l=1 \\ (l \neq k)}}^n \left[1 - \frac{\mu^2}{(\lambda_k - \lambda_l)^2}\right]^{1/2} \left[1 - \frac{\mu^2}{(\lambda_k + \lambda_l)^2}\right]^{1/2} \end{aligned} \quad (\text{A.4})$$

where

$$V_0(\lambda; u, v, \mu) = \frac{uv}{\mu^2} \prod_{k=1}^n \left[1 - \frac{\mu^2}{\lambda_k^2}\right] - \frac{uv}{\mu^2}. \quad (\text{A.5})$$

The limiting Hamiltonian H_0 can be recognised as the action-angle dual of the standard trigonometric BC_n Hamiltonian. The latter was derived in [25] by reduction of the cotangent bundle of $T^*\mathbb{U}(2n)$, and was denoted there by \tilde{H}^0 . Concretely, the correspondence with the notations used in equation (1.4) of [25] is

$$H_0(\lambda, \theta; u, v, 2\mu) = \tilde{H}^0(\lambda, \vartheta; \kappa, \nu, \mu) \quad (\text{A.6})$$

under the substitutions

$$u \rightarrow -\kappa, \quad v \rightarrow \nu, \quad \theta \rightarrow \vartheta. \quad (\text{A.7})$$

We note for completeness that [25] adopted the inessential condition $\nu > |\kappa| \geq 0$.

References

- [1] F. Calogero, *Solution of the one-dimensional N -body problem with quadratic and/or inversely quadratic pair potentials*, J. Math. Phys. **12**, 419-436 (1971)
- [2] B. Sutherland, *Exact results for a quantum many-body problem in one dimension*, Phys. Rev. A **4**, 2019-2021 (1971)
- [3] J. Moser, *Three integrable Hamiltonian systems connected with isospectral deformations*, Adv. Math. **16**, 197-220 (1975)

- [4] M.A. Olshanetsky and A.M. Perelomov, *Classical integrable finite-dimensional systems related to Lie algebras*, Phys. Rep. **71**, 313-400 (1981)
- [5] S.N.M. Ruijsenaars and H. Schneider, *A new class of integrable systems and its relation to solitons*, Ann. Phys. **170**, 370-405 (1986)
- [6] J.F. van Diejen, *Deformations of Calogero-Moser systems*, Theor. Math. Phys. **99**, 549-554 (1994); arXiv:solv-int/9310001
- [7] N. Nekrasov, *Infinite-dimensional algebras, many-body systems and gauge theories*, pp. 263-299 in: Moscow Seminar in Mathematical Physics, AMS Transl. Ser. 2, Vol. **191**, American Mathematical Society, 1999
- [8] S.N.M. Ruijsenaars, *Systems of Calogero-Moser type*, pp. 251-352 in: Proceedings of the 1994 CRM-Banff Summer School Particles and Fields, Springer, 1999
- [9] J.F. van Diejen and L. Vinet (Editors), *Calogero-Moser-Sutherland Models*, Springer, 2000
- [10] B. Sutherland, *Beautiful Models*, Word Scientific, 2004
- [11] A.P. Polychronakos, *Physics and mathematics of Calogero particles*, J. Phys. A: Math. Gen. **39**, 12793-12845 (2006); arXiv:hep-th/0607033
- [12] P. Etingof, *Calogero-Moser Systems and Representation Theory*, European Mathematical Society, 2007
- [13] S.N.M. Ruijsenaars, *Action-angle maps and scattering theory for some finite-dimensional integrable systems. I. The pure soliton case*, Commun. Math. Phys. **115**, 127-165 (1988)
- [14] S.N.M. Ruijsenaars, *Action-angle maps and scattering theory for some finite-dimensional integrable systems III. Sutherland type systems and their duals*, Publ. RIMS **31**, 247-353 (1995)
- [15] D. Kazhdan, B. Kostant and S. Sternberg, *Hamiltonian group actions and dynamical systems of Calogero type*, Comm. Pure Appl. Math. **XXXI**, 481-507 (1978)
- [16] V. Fock, A. Gorsky, N. Nekrasov and V. Rubtsov, *Duality in integrable systems and gauge theories*, JHEP **07**, 028 (2000); arXiv:hep-th/9906235
- [17] L. Fehér and C. Klimčík, *Poisson-Lie interpretation of trigonometric Ruijsenaars duality*, Commun. Math. Phys. **301**, 55-104 (2011); arXiv:0906.4198 [math-ph]
- [18] I. Marshall, *A new model in the Calogero-Ruijsenaars family*, Commun. Math. Phys. **338**, 563-587 (2015); arXiv:1311.4641 [math-ph]
- [19] L. Fehér and T.F. Görbe, *On a Poisson-Lie deformation of the $BC(n)$ Sutherland system*, Nucl. Phys. B **901**, 85-114 (2015); arXiv:1508.04991 [math-ph]
- [20] M.A. Semenov-Tian-Shansky, *Dressing transformations and Poisson groups actions*, Publ. RIMS **21**, 1237-1260 (1985)
- [21] L. Fehér and T.F. Görbe, *The full phase space of a model in the Calogero-Ruijsenaars family*, J. Geom. Phys. **115**, 139-149 (2017); arXiv:1603.02877 [math-ph]

- [22] I. Marshall, *Spectral parameter dependent Lax pairs for systems of Calogero-Moser type*, Lett. Math. Phys. **107**, 619-642 (2017)
- [23] L. Fehér and B.G. Puztai, *A class of Calogero type reductions of free motion on a simple Lie group*, Lett. Math. Phys. **79**, 263-277 (2007); [arXiv:math-ph/0609085](#)
- [24] B.G. Puztai, *The hyperbolic $BC(n)$ Sutherland and the rational $BC(n)$ Ruijsenaars-Schneider-van Diejen models: Lax matrices and duality*, Nucl. Phys. B **856**, 528-551 (2012); [arXiv:1109.0446 \[math-ph\]](#)
- [25] L. Fehér and T.F. Görbe, *Duality between the trigonometric BC_n Sutherland system and a completed rational Ruijsenaars-Schneider-van Diejen system*, J. Math. Phys. **55**, 102704 (2014); [arXiv:1407.2057 \[math-ph\]](#)
- [26] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian (with an Appendix by I. G. Macdonald)*, Invent. Math. **133**, 141 (1998)
- [27] M.A. Semenov-Tian-Shansky, *Integrable systems: an r-matrix approach*, Kyoto preprint RIMS-1650, 2008, www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1650.pdf
- [28] J-H. Lu, *Momentum mappings and reduction of Poisson actions*, pp. 209-226 in: Symplectic Geometry, Groupoids, and Integrable Systems, Springer, 1991
- [29] L. Fehér and I. Marshall, preprint in preparation
- [30] B.G. Puztai and T.F. Görbe, *Lax representation of the hyperbolic van Diejen dynamics with two coupling parameters*, to appear in Commun. Math. Phys., [arXiv:1603.06710 \[math-ph\]](#)
- [31] J.F. van Diejen and E. Emsiz, *Spectrum and eigenfunctions of the lattice hyperbolic Ruijsenaars-Schneider system with exponential Morse term*, Annales Henri Poincaré **17**, 1615-1629 (2016); [arXiv:1508.03829 \[math-ph\]](#)