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Some applications of Retkes' identity

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Abstract. We present some formulas for certain numeric sums related to the Riemann zeta function. The main tool used in our investigation is Retkes' identity. We get a formula for $\zeta(3)$ with the Euler beta function in it.

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1. Introduction. Retkes proved in [1] the following interesting theorem as an extension of the Hermite–Hadamard inequality.

Theorem 1.1. *Suppose that $\infty \leq a < b \leq \infty$, and let $f : [a, b) \to \mathbb{R}$ be a convex function, $x_i \in (a, b)$, $i = 1, \ldots, n$, such that $x_i \neq x_j$ if $1 \leq i < j \leq n$. Then the following inequality holds:*

$$\sum_{k=1}^{n} \frac{F^{(n-1)}(x_k)}{\Pi_k(x_1, \ldots, x_n)} \leq \frac{1}{n!} \sum_{k=1}^{n} f(x_k)$$

where $F^{(j)}$ is the jth iterated integral of f and

$$\Pi_k(x_1, \ldots, x_n) = \prod_{\substack{j=1 \\ j \neq k}}^{n} (x_k - x_j).$$

In the concave case "$\leq$" is changed to "$\geq$".

Moreover, he showed some consequences and applications of Theorem 1.1, see [1,2]. We need the following identity [1].

Birkhäuser

Corollary 1.2. *If $x_k \neq 0$, $k = 1, \dots, n$ and $x_i \neq x_j$ if $1 \le i < j \le n$, then*

$$\sum_{k=1}^{n} \frac{1}{x_k} = (-1)^{n-1} \prod_{k=1}^{n} x_k \sum_{k=1}^{n} \frac{1}{x_k^2 \Pi_k(x_1, \dots, x_n)} .$$

2. Application. We want to investigate the application of Corollary 1.2 for certain sums. First, for the sum

$$\sum_{k=1}^{n} \frac{1}{k} ,$$

using the fact

$$\Pi_k(1, 2, \dots, n) = (-1)^{n-k}(k-1)!(n-k)! ,$$

one can easily get the well-known formula [3, p. 5]

Formula 2.1.

$$\sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} \binom{n}{k} .$$

Now apply Corollary 1.2 for the sum

$$\sum_{k=1}^{n} \frac{1}{2k-1} .$$

We have that

$$\Pi_k(1, 3, \dots, 2n-1) = 2^{n-1}(-1)^{n-k}(k-1)!(n-k)! .$$

Hence

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{2k-1} &= (-1)^{n-1} \prod_{k=1}^{n} (2k-1) \sum_{k=1}^{n} \frac{1}{(2k-1)^2\, 2^{n-1}(-1)^{n-k}(k-1)!(n-k)!} \\ &= \sum_{k=1}^{n} (-1)^{k-1} \frac{(2n)!}{(2k-1)^2\, 2^{2n-1} n!(k-1)!(n-k)!} , \end{aligned}$$

and we obtain

Formula 2.2.

$$\sum_{k=1}^{n} \frac{1}{2k-1} = \binom{2n}{n} \sum_{k=1}^{n} (-1)^{k-1} \frac{k}{(2k-1)^2\, 2^{2n-1}} \binom{n}{k} .$$

Consider the sum

$$\sum_{k=1}^{n} \frac{1}{k^2} .$$

We calculate that

$$\Pi_k(1,4,\ldots,n^2) = \Pi_k(1,2,\ldots,n)\,\frac{1}{2k}\prod_{j=1}^{n}(k+j)$$
$$= \frac{(-1)^{n-k}(k-1)!(n-k)!(n+k)!}{2k\cdot k!}$$

and

$$\sum_{k=1}^{n}\frac{1}{k^2} = (-1)^{n-1}\prod_{k=1}^{n}k^2\sum_{k=1}^{n}\frac{2k\cdot k!}{k^4(-1)^{n-k}(k-1)!(n-k)!(n+k)!}$$
$$= 2\sum_{k=1}^{n}(-1)^{k-1}\frac{(n!)^2}{k^2(n-k)!(n+k)!}\,.$$

Hence we proved

Formula 2.3.

$$\sum_{k=1}^{n}\frac{1}{k^2} = 2\sum_{k=1}^{n}(-1)^{k-1}\frac{\binom{n}{k}}{k^2\binom{n+k}{n}} = 2\sum_{k=1}^{n}(-1)^{k-1}\frac{\binom{2n}{n+k}}{k^2\binom{2n}{n}}\,.$$

Let us take a look at

$$\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\,.$$

Now

$$\Pi_k(1,3^2,\ldots,(2n-1)^2) = \Pi_k(1,3,\ldots,2n-1)\,\frac{1}{2(2k-1)}\prod_{j=1}^{n}2(k+j-1)$$
$$= \frac{2^{n-1}(-1)^{n-k}(k-1)!(n-k)!\,2^n(n+k-1)!}{2(2k-1)\cdot(k-1)!}\,,$$

thus

$$\sum_{k=1}^{n}\frac{1}{(2k-1)^2} = (-1)^{n-1}\prod_{k=1}^{n}(2k-1)^2$$
$$\times\sum_{k=1}^{n}\frac{2(2k-1)}{(2k-1)^4 2^{n-1}(-1)^{n-k}(n-k)!\,2^n(n+k-1)!}$$
$$= \sum_{k=1}^{n}(-1)^{k-1}\frac{((2n)!)^2}{(2k-1)^3 2^{2(2n-1)}(n!)^2(n-k)!(n+k-1)!}\,.$$

We get

Formula 2.4.

$$\sum_{k=1}^{n}\frac{1}{(2k-1)^2} = \binom{2n}{n}\sum_{k=1}^{n}(-1)^{k-1}\frac{n+k}{(2k-1)^3 2^{2(2n-1)}}\binom{2n}{n+k}\,.$$

It is interesting to consider the infinite series converging to $\zeta(3)$,

$$\zeta(3) = \sum_{k=1}^{\infty}\frac{1}{k^3}\,.$$

For the partial sums $\sum_{k=1}^{n} \frac{1}{k^3}$, we need

$$\Pi_k(1,8,\ldots,n^3) = \Pi_k(1,2,\ldots,n)\, \frac{1}{3k^2} \prod_{j=1}^{n}(k^2+kj+j^2)$$
$$= (-1)^{n-k} k!(n-k)!\, \frac{1}{3k^3} \prod_{j=1}^{n}(k^2+kj+j^2)\,.$$

Then we have

$$\sum_{k=1}^{n} \frac{1}{k^3} = (-1)^{n-1} \prod_{k=1}^{n} k^3 \sum_{k=1}^{n} \frac{3k^3}{k^6(-1)^{n-k} k!(n-k)! \prod_{j=1}^{n}(k^2+kj+j^2)}$$
$$= 3\sum_{k=1}^{n} (-1)^{k-1} \frac{(n!)^3}{k^3 k!(n-k)! \prod_{j=1}^{n}(k^2+kj+j^2)}$$

and

Formula 2.5.

$$\sum_{k=1}^{n} \frac{1}{k^3} = 3\sum_{k=1}^{n} (-1)^{k-1} \frac{\binom{n}{k}(n!)^2}{k^3 \prod_{j=1}^{n}(k^2+kj+j^2)}\,.$$

Before our formula for $\zeta(3)$, we study the terms in Formula 2.5. First one can see that for an arbitrarily fixed k,

$$\frac{\binom{n}{k}(n!)^2}{\prod_{j=1}^{n}(k^2+kj+j^2)} \to k \cdot B\left(\frac{1}{2}k + \frac{\sqrt{3}}{2}ki, \frac{1}{2}k - \frac{\sqrt{3}}{2}ki\right) \quad \text{as} \quad n\to\infty \tag{2.1}$$

where $B(x,y)$ is the Euler beta function [3, p. 909]

$$B(x,y) = \frac{x+y}{xy} \prod_{j=1}^{\infty} \frac{j(x+y+j)}{(x+j)(y+j)} \quad (x,y \neq 0,-1,\ldots).$$

Indeed, for a fixed k,

$$\frac{\binom{n}{k}(n!)^2}{\prod_{j=1}^{n}(k^2+kj+j^2)} = k \cdot \frac{k}{k^2} \prod_{j=1}^{n} \frac{j(j+k)}{k^2+kj+j^2} \prod_{j=1}^{k} \frac{n-j+1}{n+j} \to kB(x_k,y_k)$$

as $n\to\infty$, where x_k and y_k are such that $x_k+y_k=k$ and $x_k y_k = k^2$. Second, for any n and $k \le n$, we have

$$\frac{\binom{n}{k}(n!)^2}{\prod_{j=1}^{n}(k^2+kj+j^2)} = \prod_{j=1}^{n} \tfrac{j(j+k)}{k^2+kj+j^2} \prod_{j=1}^{k} \tfrac{n-j+1}{n+j} \le 1. \tag{2.2}$$

Then we will prove the following formula.

Formula 2.6.

$$\zeta(3) = 3\sum_{k=1}^{\infty} (-1)^{k-1} \frac{B(\frac{1}{2}k + \frac{\sqrt{3}}{2}ki, \frac{1}{2}k - \frac{\sqrt{3}}{2}ki)}{k^2}\,.$$

Proof. Set $\varepsilon > 0$ arbitrarily. There exists $m(\varepsilon)$ for which any $n > m(\varepsilon)$ satisfies

$$\sum_{k=n}^{\infty} \frac{1}{k^3} < \varepsilon.$$

Now fix $n > m(\varepsilon)$ arbitrarily. By (2.1), there exists an $N \geq n$ such that

$$\left| \frac{\binom{N}{k}(N!)^2}{\prod_{j=1}^{N}(k^2+kj+j^2)} - k \cdot B\left(\frac{1}{2}k + \frac{\sqrt{3}}{2}ki, \frac{1}{2}k - \frac{\sqrt{3}}{2}ki\right) \right| < \varepsilon$$

for any $1 \leq k \leq n$. Then using (2.2), we get

$$\left| 3\sum_{k=1}^{n}(-1)^{k-1}\frac{B(\frac{1}{2}k+\frac{\sqrt{3}}{2}ki, \frac{1}{2}k-\frac{\sqrt{3}}{2}ki)}{k^2} - 3\sum_{k=1}^{N}(-1)^{k-1}\frac{\binom{N}{k}(N!)^2}{k^3\prod_{j=1}^{N}(k^2+kj+j^2)} \right|$$
$$< 3\sum_{k=1}^{n}\frac{\varepsilon}{k^3} + 3\sum_{k=n+1}^{N}\frac{1}{k^3} < (3\zeta(3)+3)\varepsilon.$$

Moreover, by Formula 2.5,

$$\left| 3\sum_{k=1}^{N}(-1)^{k-1}\frac{\binom{N}{k}(N!)^2}{k^3\prod_{j=1}^{N}(k^2+kj+j^2)} - \zeta(3) \right| = \left| \sum_{k=1}^{N}\frac{1}{k^3} - \zeta(3) \right| < \varepsilon,$$

hence

$$\left| 3\sum_{k=1}^{n}(-1)^{k-1}\frac{B\left(\frac{1}{2}k+\frac{\sqrt{3}}{2}ki, \frac{1}{2}k-\frac{\sqrt{3}}{2}ki\right)}{k^2} - \zeta(3) \right| < (3\zeta(3)+4)\varepsilon,$$

which is the required result. □

It is a natural question what happens if we apply Corollary 1.2 for the sums

$$\sum_{k=1}^{n}\frac{1}{k^r}$$

if r is an integer greater than 3, since these sums are the partial sums of the appropriate Riemann zeta function values

$$\zeta(r) = \sum_{k=1}^{\infty}\frac{1}{k^r}.$$

We calculate like we did in the previous cases.

$$\Pi_k(1, 2^r, \ldots, n^r) = \Pi_k(1, 2, \ldots, n)\,\frac{1}{rk^{r-1}}\prod_{j=1}^{n}\left(\sum_{\ell=1}^{r} j^{\ell-1}k^{r-\ell}\right)$$
$$= (-1)^{n-k}k!(n-k)!\,\frac{1}{rk^r}\prod_{j=1}^{n}\left(\sum_{\ell=1}^{r} j^{\ell-1}k^{r-\ell}\right).$$

Then we have

Formula 2.7.

$$\sum_{k=1}^{n}\frac{1}{k^r}=(-1)^{n-1}\prod_{k=1}^{n}k^r\sum_{k=1}^{n}\frac{rk^r}{k^{2r}(-1)^{n-k}k!(n-k)!\prod_{j=1}^{n}\left(\sum_{\ell=1}^{r}j^{\ell-1}k^{r-\ell}\right)}$$
$$=r\sum_{k=1}^{n}(-1)^{k-1}\frac{\binom{n}{k}}{k^r\prod_{j=1}^{n}\left(\sum_{\ell=0}^{r-1}(j/k)^{\ell}\right)}=r\sum_{k=1}^{n}(-1)^{k-1}a(n,k,r)\,.$$

To produce an identity for $\zeta(r)$ similar to Formula 2.6, with a similar argumentation as in the proof of Formula 2.6, we can write

$$\zeta(r)=r\sum_{k=1}^{n}(-1)^{k-1}a(k,r)\,,$$

where $a(k,r)=\lim_{n\to\infty}a(n,k,r)$. For a fair result we need to calculate the values $a(k,r)$. We can do this by using the software *Mathematica* [4]. The more interesting case is when r is odd since for even r, the value $\zeta(r)$ is well-studied. After calculating the limits $a(k,r)$ for $r=5,7,9$, we conjecture the following formula for odd integers $r>3$:

$$\zeta(r)=r\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\prod_{j=1}^{r-1}\Gamma\left(1+(-1)^{j-1}(-1)^{j/r}k\right)}{k^r\,\Gamma(1+k)}\,.$$

where $\Gamma(z)$ is the Euler gamma function [3].

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