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# ON $\Lambda^r$–STRONG CONVERGENCE OF NUMERICAL SEQUENCES AND FOURIER SERIES

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*Abstract.* We prove theorems of interest about the recently given $\Lambda^r$-strong convergence. The main goal is to extend the results of F. Móricz regarding the $\Lambda$-strong convergence of numerical sequences and Fourier series.

## 1. Introduction

Throughout this paper let $\Lambda = \{\lambda_k : k = 0, 1, \ldots\}$ be a non-decreasing sequence of positive numbers tending to $\infty$. The concept of $\Lambda$-strong convergence was introduced in [2]. We say, that a sequence $S = \{s_k : k = 0, 1, \ldots\}$ of complex numbers converges $\Lambda$-strongly to a complex number $s$ if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} |\lambda_k(s_k - s) - \lambda_{k-1}(s_{k-1} - s)| = 0$$

with the agreement $\lambda_{-1} = s_{-1} = 0$.

It is useful to note that $\Lambda$-strong convergence is an intermediate notion between bounded variation and ordinary convergence.

The following generalization was suggested recently in [1]. Throughout this paper, we assume that $r \geqslant 2$ is an integer. A sequence $S = \{s_k\}$ of complex numbers is said to converge $\Lambda^r$-strongly to a complex number $s$ if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} |\lambda_k(s_k - s) - \lambda_{k-r}(s_{k-r} - s)| = 0$$

with the agreement $\lambda_{-1} = \ldots = \lambda_{-r} = s_{-1} = \ldots = s_{-r} = 0$.

It was seen that these $\Lambda^r$-convergence notions are intermediate notions between $\Lambda$-strong convergence and ordinary convergence. The following two basic results were introduced in [1] as Propositions 1 and 2, synthesized from [1, Lemmas 1 and 2].

LEMMA 1. *A sequence $S$ converges $\Lambda^r$-strongly to a number $s$ if and only if*

(i) *$S$ converges to $s$ in the ordinary sense, and*

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(ii) $\lim\limits_{n\to\infty} \dfrac{1}{\lambda_n} \sum\limits_{k=r}^{n} \lambda_{k-r} |s_k - s_{k-r}| = 0.$

LEMMA 2. *A sequence $S$ converges $\Lambda^r$-strongly to a number $s$ if and only if*

$$\sigma_n := \frac{1}{\lambda_n} \sum_{\substack{0 \leqslant k \leqslant n \\ r|n-k}} (\lambda_k - \lambda_{k-r}) s_k$$

*converges to $s$ in the ordinary sense and condition* (ii) *is satisfied.*

## 2. Results on numerical sequences

Denote by $c^r(\Lambda)$ the class of $\Lambda^r$-strong convergent sequences $S = \{s_k\}$ of complex numbers. Obviously, $c^r(\Lambda)$ is a linear space. Let

$$\|S\|_{c^r(\Lambda)} := \sup_{n \geqslant 0} \frac{1}{\lambda_n} \sum_{k=0}^{n} |\lambda_k s_k - \lambda_{k-r} s_{k-r}|,$$

and consider the well-known norms

$$\|S\|_\infty := \sup_{k \geqslant 0} |s_k|, \quad \|S\|_{\mathrm{bv}} := \sum_{k=0}^{\infty} |s_k - s_{k-1}|.$$

It is easy to see that $\|.\|_{c^r(\Lambda)}$ is also a norm on $c^r(\Lambda)$.

Moreover, one can easily obtain the inequality

$$\sum_{k=0}^{n} |\lambda_k s_k - \lambda_{k-r} s_{k-r}| \leqslant r \sum_{k=0}^{n} |\lambda_k s_k - \lambda_{k-1} s_{k-1}|$$

and the equality

$$s_k = \frac{1}{\lambda_n} \sum_{\substack{0 \leqslant k \leqslant n \\ r|n-k}} (\lambda_k s_k - \lambda_{k-r} s_{k-r}).$$

These together imply the following result.

PROPOSITION 1. *For every sequence $S = \{s_k\}$ of complex numbers we have*

$$\|S\|_\infty \leqslant \|S\|_{c^r(\Lambda)} \leqslant r\|S\|_{c(\Lambda)} \leqslant 2r\|S\|_{\mathrm{bv}}.$$

*As a consequence,* $\mathrm{bv} \subset c(\Lambda) \subset c^r(\Lambda) \subset c$.

It was seen in [2] that $c(\Lambda)$ endowed with the norm $\|.\|_{c(\Lambda)}$ is a Banach space. A similar results holds for $c^r(\Lambda)$.

THEOREM 1. *The class $c^r(\Lambda)$ endowed with the norm $\|.\|_{c^r(\Lambda)}$ is a Banach space.*

*Proof.* With an analogous argument to the proof of [2, Theorem 1], we can get the required completeness of $c^r(\Lambda)$. The only needed modifications are

$$\frac{1}{\lambda_n}\sum_{k=0}^{n}|\lambda_k(s_{\ell k}-s_k)-\lambda_{k-r}(s_{\ell,k-r}-s_{k-r})|\leqslant\|S_\ell-S\|_\infty\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k+\lambda_{k-r})\leqslant\varepsilon$$

and

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k=0}^{n}&|\lambda_k(s_{jk}-s_k)-\lambda_{k-r}(s_{j,k-r}-s_{k-r})|\\ &\leqslant\frac{1}{\lambda_n}\sum_{k=0}^{n}|\lambda_k(s_{jk}-s_{\ell k})-\lambda_{k-r}(s_{j,k-r}-s_{\ell,k-r})|\\ &\quad+\frac{1}{\lambda_n}\sum_{k=0}^{n}|\lambda_k(s_{\ell k}-s_k)-\lambda_{k-r}(s_{\ell,k-r}-s_{k-r})|\\ &\leqslant\|S_j-S_\ell\|_{c^r(\Lambda)}+\varepsilon\leqslant 2\varepsilon\end{aligned}$$

for large enough $\ell$ and $j$. □

Now that we saw that $c^r(\Lambda)$ is a Banach space, we show that it has a Schauder basis. In fact, putting

$$F^{(j)}:=(0,0,\dots,0,\overbrace{1}^{j},0,0,\dots,0,\overbrace{1}^{j+r},0,0,\dots,0,\overbrace{1}^{j+2r},\dots),$$

$j=0,1,\dots,$ clearly each $F^{(j)}\in c^r(\Lambda)$.

THEOREM 2. *$\{F^{(j)}:j=0,1,\dots\}$ is a basis in $c^r(\Lambda)$.*

*Proof. Existence.* We will show that if $S=\{s_k\}$ is a $\Lambda^r$-strongly convergent sequence, then

$$\lim_{m\to\infty}\|S-\sum_{j=0}^{m}(s_j-s_{j-r})F^{(j)}\|_{c^r(\Lambda)}=0. \tag{1}$$

Since

$$\begin{aligned}&S-\sum_{j=0}^{m}(s_j-s_{j-r})F^{(j)}\\ &\quad=(0,0,\dots,\overbrace{0}^{m},\overbrace{s_{m+1}-s_{m-r+1}}^{m+1},s_{m+2}-s_{m-r+2},\dots,\overbrace{s_{m+ar+b}-s_{m-r+b}}^{m+ar+b},\dots),\end{aligned}$$

where $0 \leqslant a$, $1 \leqslant b \leqslant r$, by definition,

$$
\begin{aligned}
&\|S-\sum_{j=0}^{m}(s_j-s_{j-r})F^{(j)}\|_{c^r(\Lambda)}\\
&\quad=\sup_{n\geqslant 1}\frac{1}{\lambda_{m+n}}\Big(\sum_{\substack{1\leqslant b\leqslant r\\ b\leqslant n}}\lambda_{m+b}|(s_{m+b}-s_{m-r+b}|\\
&\qquad\qquad+\sum_{a=1}^{[n/r]}\sum_{\substack{1\leqslant b\leqslant r\\ ar+b\leqslant n}}|\lambda_{m+ar+b}(s_{m+ar+b}-s_{m-r+b})\\
&\qquad\qquad\qquad-\lambda_{m+ar-r+b}(s_{m+ar-r+b}-s_{m-r+b})|\Big)\\
&\quad\leqslant\sup_{n\geqslant 1}\frac{1}{\lambda_{m+n}}\Big(\sum_{a=0}^{[n/r]}\sum_{\substack{1\leqslant b\leqslant r\\ ar+b\leqslant n}}(\lambda_{m+ar+b}-\lambda_{m+ar-r+b})|s_{m+ar+b}-s_{m-r+b}|\\
&\qquad\qquad+\sum_{a=0}^{[n/r]}\sum_{\substack{1\leqslant b\leqslant r\\ ar+b\leqslant n}}\lambda_{m+ar-r+b}|s_{m+ar+b}-s_{m+ar-r+b}|\Big)\\
&\quad\leqslant r\sup_{j,k>m-r}|s_j-s_k|+\sup_{n\geqslant m+1}\frac{1}{\lambda_n}\sum_{k=m+1}^{n}\lambda_{k-r}|s_k-s_{k-r}|.
\end{aligned}
$$

Applying Proposition 1 and Lemma 1, respectively, results in (1) to be proved.

*Uniqueness.* It can be proved in basically the same way as it was seen in the proof of [2, Theorem 2]. □

### 3. Results on Fourier series: $C$-metric

Denote by $C$ the Banach space of the $2\pi$ periodic complex-valued continuous functions endowed with the norm $\|f\|_C := \max_t |f(t)|$. Let

$$\frac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k(f)\cos kt+b_k(f)\sin kt) \tag{2}$$

be the Fourier series of a function $f\in C$ with the usual notation $s_k(f)=s_k(f,t)$ for the $k$th partial sum of the series (2). Denote by $U$, $A$, and $S(\Lambda)$, respectively, the classes of functions $f\in C$ whose Fourier series converges uniformly, converges absolutely and converges uniformly $\Lambda$-strongly on $[0,2\pi)$, endowed with the usual norms, see [2].

A function $f\in C$ belongs to $S(\Lambda^r)$ if

$$\lim_{n\to\infty}\left\|\frac{1}{\lambda_n}\sum_{k=0}^{n}|\lambda_k(s_k(f)-f)-\lambda_{k-r}(s_{k-r}(f)-f)|\right\|_C=0.$$

Set the norm

$$\|f\|_{S(\Lambda^r)} := \sup_{n\geqslant 0}\left\|\frac{1}{\lambda_n}\sum_{k=0}^{n}|\lambda_k s_k(f)-\lambda_{k-r}s_{k-r}(f)|\right\|_C,$$

which is finite for every for $S(\Lambda^r)$ since

$$\|f\|_{S(\Lambda^r)} \leqslant \|f\|_C + \sup_{n\geqslant 0}\left\|\frac{1}{\lambda_n}\sum_{k=0}^{n}|\lambda_k(s_k(f)-f)-\lambda_{k-r}(s_{k-r}(f)-f)|\right\|_C.$$

The norm inequalities corresponding to the ones in Proposition 1 are formulated below.

PROPOSITION 2. *For every function $f\in C$ we have*

$$\|f\|_U \leqslant \|f\|_{S(\Lambda^r)} \leqslant r\|f\|_{S(\Lambda)} \leqslant 2r\|f\|_A.$$

*As a consequence, $A\subset S(\Lambda)\subset S(\Lambda^r)\subset U$.*

The following results are the counterparts to Lemmas 1 and 2 and Theorems 1 and 2, respectively. We omit the details of the analogous proofs, except for Theorem 4.

LEMMA 3. *A function $f$ belongs to $S(\Lambda^r)$ if and only if*

(iii) $\lim\limits_{k\to\infty}\|s_k(f)-f\|_C=0$, *and*

(iv) $\lim\limits_{n\to\infty}\left\|\frac{1}{\lambda_n}\sum_{k=r}^{n}\lambda_{k-r}|s_k(f)-s_{k-r}(f)|\right\|_C=0.$

LEMMA 4. *A function $f$ belongs to $S(\Lambda^r)$ if and only if*

(iii') $\lim\limits_{n\to\infty}\|\sigma_n(f)-f\|_C=0$

*and condition* (iv) *is satisfied, where*

$$\sigma_n(f)=\sigma_n(f,t):=\frac{1}{\lambda_n}\sum_{\substack{0\leqslant k\leqslant n\\ r|n-k}}(\lambda_k-\lambda_{k-r})s_k(f,t).$$

THEOREM 3. *The set $S(\Lambda^r)$ endowed with the norm $\|.\|_{S(\Lambda^r)}$ is a Banach space.*

THEOREM 4. *If $f\in S(\Lambda^r)$, then*

$$\lim_{m\to\infty}\|s_m(f)-f\|_{S(\Lambda^r)}=0. \tag{3}$$

*Proof.* Since the sequence of partial sums of the Fourier series of the difference $f - s_m(f)$ is

$$(0, 0, \ldots, \overbrace{0}^{m}, \overbrace{s_{m+1}(f) - s_m(f)}^{m+1}, s_{m+2}(f) - s_m(f), \ldots),$$

then

$$\begin{aligned}
&\|s_m(f) - f\|_{S(\Lambda^r)} \\
&\quad = \sup_{n \geqslant 1} \frac{1}{\lambda_{m+n}} \Bigg\| \sum_{\substack{1 \leqslant b \leqslant r \\ b \leqslant n}} \lambda_{m+b} |(s_{m+b}(f) - s_m(f)| \\
&\qquad\qquad + \sum_{a=1}^{[n/r]} \sum_{\substack{1 \leqslant b \leqslant r \\ ar+b \leqslant n}} |\lambda_{m+ar+b}(s_{m+ar+b}(f) - s_m(f)) \\
&\qquad\qquad\qquad - \lambda_{m+ar-r+b}(s_{m+ar-r+b}(f) - s_m(f))| \Bigg\|_C \\
&\quad \leqslant \sup_{n \geqslant 1} \frac{1}{\lambda_{m+n}} \Bigg\| \sum_{a=0}^{[n/r]} \sum_{\substack{1 \leqslant b \leqslant r \\ ar+b \leqslant n}} (\lambda_{m+ar+b} - \lambda_{m+ar-r+b}) |s_{m+ar+b}(f) - s_m(f)| \\
&\qquad\qquad + \sum_{a=0}^{[n/r]} \sum_{\substack{1 \leqslant b \leqslant r \\ ar+b \leqslant n}} \lambda_{m+ar-r+b} |s_{m+ar+b}(f) - s_{m+ar-r+b}(f)| \Bigg\|_C \\
&\quad \leqslant r \sup_{j,k > m-r} \|s_j(f) - s_k(f)\|_C + \sup_{n \geqslant m+1} \Bigg\| \frac{1}{\lambda_n} \sum_{k=m+1}^{n} \lambda_{k-r} |s_k(f) - s_{k-r}(f)| \Bigg\|_C,
\end{aligned}$$

where $0 \leqslant a$, $1 \leqslant b \leqslant r$. Applying Proposition 2 and Lemma 3, respectively, results in (3) to be proved. $\square$

In the following, our goal is to extend the well-known Denjoy–Luzin theorem presented below (see [3, p. 232]).

THEOREM 5. (Theorem of Denjoy–Luzin) *If*

$$\sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \tag{4}$$

*converges absolutely for $t$ belonging to a set $A$ of positive measure, then* $\sum_{k=1}^{\infty} (|a_k| + |b_k|)$ *converges.*

This theorem was extended for $\Lambda$-strongly convergent trigonometric series by Móricz in [2].

THEOREM 6. *If the $n$th partial sums $s_n(t)$ of the series (4) converge $\Lambda$-strongly for $t$ belonging to a set $A$ of positive measure or of second category, then*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=1}^{n} \lambda_{k-1}(|a_k|+|b_k|) = 0. \tag{5}$$

*Consequently, if $f \in C$ and the $n$th partial sums $s_n(f,t)$ of the Fourier series (2) converge uniformly $\Lambda$-strongly to $f(t)$ everywhere, then coefficients $a_k = a_k(f)$ and $b_k = b_k(f)$ satisfy (5).*

First, we extend Theorem 5 for single sine and cosine series.

THEOREM 7. *If*

$$\sum_{k=1}^{\infty} |a_{2k-1}\cos(2k-1)t + a_{2k}\cos 2kt| \quad \textit{and} \quad \sum_{k=1}^{\infty} |a_{2k-1}\sin(2k-1)t + a_{2k}\sin 2kt|$$

*converge for $t$ belonging to a set $A$ of positive measure, then $\sum_{k=1}^{\infty} |a_k|$ converges.*

*Proof.* We follow the proof of the Denjoy–Luzin theorem as in [3, pp. 232] with necessary modifications. We calculate

$$\begin{aligned} &a_{2k-1}\cos(2k-1)t + a_{2k}\cos 2kt \\ &\qquad = (a_{2k-1} + a_{2k}\cos t)\cos(2k-1)t - (a_{2k}\sin t)\sin(2k-1)t \end{aligned}$$

and

$$\begin{aligned} &a_{2k-1}\sin(2k-1)t + a_{2k}\sin 2kt \\ &\qquad = (a_{2k-1} + a_{2k}\cos t)\sin(2k-1)t + (a_{2k}\sin t)\cos(2k-1)t, \end{aligned}$$

whence

$$a_{2k-1}\cos(2k-1)t + a_{2k}\cos 2kt = \rho_k(t)\cos((2k-1)t + f_k(t))$$

and

$$a_{2k-1}\sin(2k-1)t + a_{2k}\sin 2kt = \rho_k(t)\sin((2k-1)t + f_k(t))$$

where

$$\rho_k(t) = \sqrt{a_{2k-1}^2 + a_{2k}^2 + 2a_{2k-1}a_{2k}\cos t}$$

and $f_k(t)$ is from

$$\cos f_k(t) = \frac{a_{2k-1} + a_{2k}\cos t}{\rho_k(t)}, \qquad \sin f_k(t) = \frac{a_{2k}\sin t}{\rho_k(t)}.$$

Now, we need that

$$\rho_k(t) \geqslant C(|a_{2k-1}| + |a_{2k}|) \tag{6}$$

is satisfied on a set $E \subseteq A$ of positive measure where the constant $C$ is independent of $k$ and $t$. Inequality (6) can be obtained from

$$(1-C^2)(a_{2k-1}^2+a_{2k}^2) \geqslant 2|a_{2k-1}a_{2k}|(C^2+|\cos t|), \tag{7}$$

since it implies

$$\rho_k^2(t) \geqslant a_{2k-1}^2+a_{2k}^2-2|a_{2k-1}a_{2k}\cos t| \geqslant C^2(|a_{2k-1}|+|a_{2k}|)^2.$$

Hence we just need to define $C$ small enough so that the set $E \subseteq A$ on which

$$|\cos t| \leqslant 1-2C^2$$

and consequently (7) and (6) hold is of positive measure. We set $C$ and thereby $E$ that way. Since $E \subseteq A$, there is a set $F \subseteq E$ of positive measure such that

$$\sum_{k=1}^{\infty} \alpha_k(t) = \sum_{k=1}^{\infty} (|a_{2k-1}\cos(2k-1)t + a_{2k}\cos 2kt| + |a_{2k-1}\sin(2k-1)t + a_{2k}\sin 2kt|)$$

is bounded on $F$, say by bound $M$. Hence we obtain the required estimation

$$\begin{aligned}
\sum_{k=1}^{\infty}(|a_{2k-1}|+|a_{2k}|) &\leqslant \frac{1}{C}\sum_{k=1}^{\infty}\int_F \rho_k(t) \\
&= \frac{1}{C}\sum_{k=1}^{\infty}\int_F \rho_k(t)(\cos^2((2k-1)t+f_k(t))+\sin^2((2k-1)t+f_k(t)))\,dt \\
&\leqslant \frac{1}{C}\sum_{k=1}^{\infty}\int_F \rho_k(t)(|\cos((2k-1)t+f_k(t))|+|\sin((2k-1)t+f_k(t))|)\,dt \\
&= \frac{1}{C}\sum_{k=1}^{\infty}\int_F \alpha_k(t)\,dt \leqslant \frac{M}{C}|F|. \quad \square
\end{aligned}$$

Second, we extend Theorem 7 for $\Lambda^2$-strong convergent sine or cosine series.

THEOREM 8. *If*

$$s_n^1(t) = \sum_{k=1}^{n} a_k \cos kt \quad \textit{and} \quad s_n^2(t) = \sum_{k=1}^{n} a_k \sin kt \tag{8}$$

*converge $\Lambda^2$-strongly for $t$ belonging to a set $A$ of positive measure, then*

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k=1}^{n} \lambda_{k-1}|a_k| = 0. \tag{9}$$

*Consequently, if $f,g \in C$ has single Fourier series $\frac{1}{2}a_0+\sum_{k=1}^{\infty} a_k\cos kt$ and $\sum_{k=1}^{\infty} a_k \sin kt$, respectively, which partial sums converge uniformly $\Lambda^2$-strongly to $f(t)$ and $g(t)$ everywhere, then coefficients $a_k$ satisfy (9).*

*Proof.* By Lemma 1 (in the second case Lemma 3 is used), $\Lambda^2$-strong convergence implies for example, in the cosine case that

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k=2}^{n}\lambda_{k-2}|s_k^1-s_{k-2}^1|=\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k=2}^{n}\lambda_{k-2}|a_{k-1}\cos(k-1)t+a_k\cos kt|=0,$$

and the proof is analogous to the one of the previous theorem, Theorem 7. $\square$

## 4. Results on Fourier series: $L^p$-metric

The results of Section 3 can be reformulated if we substitute $L^p$-metric for $C$-metric. Here and in the sequel $1\leqslant p<\infty$. Along with the usual notations let us call a function $f\in L^p$ to be in $S^p(\Lambda^r)$ if

$$\lim_{n\to\infty}\left\|\frac{1}{\lambda_n}\sum_{k=0}^{n}|\lambda_k(s_k(f)-f)-\lambda_{k-r}(s_{k-r}(f)-f)|\right\|_p=0,$$

and introduce the norm

$$\|f\|_{S^p(\Lambda^r)}:=\sup_{n\geqslant 0}\left\|\frac{1}{\lambda_n}\sum_{k=0}^{n}|\lambda_k s_k(f)-\lambda_{k-r}s_{k-r}(f)|\right\|_p,$$

which is finite for every for $S^p(\Lambda^r)$.

The norm inequalities corresponding to the ones in Proposition 2 are the following.

PROPOSITION 3. *For every function $f\in L^p$ and $r\geqslant 2$ integer we have*

$$\|f\|_{U^p}\leqslant\|f\|_{S^p(\Lambda^r)}\leqslant r\|f\|_{S^p(\Lambda)}\leqslant 2r\|f\|_A.$$

*As a consequence, $A\subset S^p(\Lambda)\subset S^p(\Lambda^r)\subset U^p$.*

The next results are analogous to Lemmas 3 and 4 and Theorems 3 and 4, respectively.

LEMMA 5. *A function $f$ belongs to $S^p(\Lambda^r)$ if and only if*

(v) $\lim_{k\to\infty}\|s_k(f)-f\|_p=0$, *and*

(vi) $\lim_{n\to\infty}\left\|\frac{1}{\lambda_n}\sum_{k=r}^{n}\lambda_{k-r}|s_k(f)-s_{k-r}(f)|\right\|_p=0.$

LEMMA 6. *A function $f$ belongs to $S^p(\Lambda^r)$ if and only if*

(v') $\lim_{n\to\infty}\|\sigma_n(f)-f\|_p=0$

*and condition* (vi) *is satisfied.*

THEOREM 9. *The set $S^p(\Lambda^r)$ endowed with the norm $\|.\|_{S^p(\Lambda^r)}$ is a Banach space.*

THEOREM 10. *If $f \in S^p(\Lambda^r)$, then*

$$\lim_{m\to\infty} \|s_m(f) - f\|_{S^p(\Lambda^r)} = 0.$$

Finally, we obtain the $L^p$-metric version of Theorem 8.

THEOREM 11. *If the sums in (8) converge $\Lambda^2$-strongly in the $L^p$-metric restricted to a set of positive measure, then (9) holds true.*

*Consequently, if $f,g \in L^p$, $1 < p < \infty$, has single Fourier series $\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kt$ and $\sum_{k=1}^{\infty} a_k \sin kt$, respectively, then the partial sums of both series converge $\Lambda^2$-strongly to $f(t)$ and $g(t)$ in the $L^p$-metric if and only if coefficients $a_k$ satisfy (9).*

*Proof.* The first statement and the necessity part of the second statement is obtained in the same way as in the proof of Theorem 7.

The sufficiency part of the second statement follows from two facts. First, by the theorem of M. Riesz [3, p. 266], (v) in Lemma 5 holds. Second, (vi) is also satisfied since

$$\lim_{n\to\infty} \left\| \frac{1}{\lambda_n} \sum_{k=2}^{n} \lambda_{k-2} |s_k(f) - s_{k-2}(f)| \right\|_p \leqslant 2 \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=1}^{n} \lambda_{k-1} |a_k| = 0. \quad \square$$

PROBLEM. Can we prove similar statements to the above proved theorems about the $\Lambda^r$-strong convergence in the case $r > 2$ as well?

## REFERENCES


[1] P. KÓRUS, *On $\Lambda^2$-strong convergence of numerical sequences revisited*, Acta Math. Hungar., **148**, 1 (2016), 222–227.

[2] F. MÓRICZ, *On $\Lambda$-strong convergence of numerical sequences and Fourier series*, Acta Math. Hungar., **54**, 3–4 (1989), 319–327.

[3] A. ZYGMUND, *Trigonometric series*, Vol. 1, Cambridge Univ. Press, 1959.


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