Acta Math. Hungar., **148** (1) (2016), 222–227
DOI: 10.1007/s10474-015-0550-5
First published online September 24, 2015

ON Λ^2-STRONG CONVERGENCE OF NUMERICAL SEQUENCES REVISITED

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(Received June 8, 2015; revised June 15, 2015; accepted June 15, 2015)

Abstract. We remark the incorrectness of some recent results concerning Λ^2-strong convergence. We give a new appropriate definition for the Λ^2-strong convergence by generalizing the original Λ-strong convergence concept given by F. Móricz.

1. Preliminaries

We are interested in the results of [1] and [2]. In [2], several results were proved using the notion of Λ-strong convergence defined there. It is essential to remind the reader of the definition. Let $\Lambda = \{\lambda_k : k = 0, 1, \ldots\}$ be a nondecreasing sequence of positive numbers tending to ∞. A sequence $X = \{x_k : k = 0, 1, \ldots\}$ of complex numbers converges Λ-strongly to a complex number x if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} \bigl| \lambda_k(x_k - x) - \lambda_{k-1}(x_{k-1} - x) \bigr| = 0$$

with the agreement $\lambda_{-1} = x_{-1} = 0$.

The two basic results proved in [2] were the following.

LEMMA M1. *A sequence X converges Λ-strongly to a number x if and only if*

(i) *X converges to x in the ordinary sense*, *and*

(ii) $$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=1}^{n} \lambda_{k-1} |x_k - x_{k-1}| = 0.$$

Key words and phrases: Λ-strong convergence, Λ^2-strong convergence, numerical sequence.
Mathematics Subject Classification: 40A05.

LEMMA M2. *A sequence X converges Λ-strongly to a number x if and only if*

$$\sigma_n := \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})x_k$$

converges to x in the ordinary sense and condition (ii) *is satisfied.*

It is useful to note that Λ-strong convergence is an intermediate notion between bounded variation and ordinary convergence.

Now we focus on [1]. The definition of Λ^2-strong convergence was introduced. Let $\Lambda = \{\lambda_k\}$ be a nondecreasing sequence of positive numbers tending to ∞ for which $\lambda_k - 2\lambda_{k-1} + \lambda_{k-2} \geqq 0$. A sequence $X = \{x_k\}$ of complex numbers converges Λ^2-strongly to a complex number x if

$$\lim_{n\to\infty}\frac{1}{\lambda_n - \lambda_{n-1}}\sum_{k=0}^{n}\big|\lambda_k(x_k - x) - 2\lambda_{k-1}(x_{k-1} - x) + \lambda_{k-2}(x_{k-2} - x)\big| = 0$$

with the agreement $\lambda_{-1} = \lambda_{-2} = x_{-1} = x_{-2} = 0$.

The first result concerning this notion was

LEMMA BM1. *A sequence X converges Λ^2-strongly to a number x if and only if condition* (i) *is satisfied and*

$$\text{(ii}'\text{)} \qquad \lim_{n\to\infty}\frac{1}{\lambda_n - \lambda_{n-1}}\sum_{k=1}^{n}\lambda_{k-1}|x_k - x_{k-1}| = 0.$$

However, the proof of Lemma BM1 is not complete in the way that only the sufficiency part was proved in [1]. The necessity part, i.e. the satisfactory of (i) and (ii′) for a Λ^2-strongly convergent sequence X was not seen. In this paper, we show that the necessity part is not true. We give a counterexample here.

COUNTEREXAMPLE. Let $x_k = \frac{1}{k+1}$ and $\lambda_k = k + 1$. It is obvious that Λ tends monotonically to ∞ with $\lambda_k - 2\lambda_{k-1} + \lambda_{k-2} \geqq 0$ satisfied. Now X converges Λ^2-strongly to 0 since

$$\lim_{n\to\infty}\frac{1}{(n+1) - n}\sum_{k=0}^{n}\left|(k+1)\frac{1}{k+1} - 2k\frac{1}{k} + (k-1)\frac{1}{k-1}\right| = \lim_{n\to\infty} 0 = 0,$$

but (ii′) is not satisfied since

$$\lim_{n\to\infty}\frac{1}{(n+1) - n}\sum_{k=1}^{n}k\left|\frac{1}{k+1} - \frac{1}{k}\right| = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k+1} = \infty.$$

In [1], the main goal was to extend the concept of Λ-strong convergence, moreover to obtain similar results as in [2]. We saw above that the first result is incorrect. If we consider the relation between Λ and Λ^2-strong convergence, we have in [1]

Proposition BM. *Let* $\sup_k \frac{\lambda_{k+1}}{\lambda_k} \leqq K$. *If* X *converges* Λ*-strongly, then it converges* Λ^2*-strongly, but the converse is not true.*

However the first statement of the proposition is proved correctly, the example for the second part is incorrect. Example 1 was given as: $x_k = \frac{2^k}{2^{k+1}+1}$ and $\lambda_k = 2^k$, and was stated to be Λ^2-strongly convergent but not Λ-strongly convergent. This is not the case, since X converges increasingly to $\frac{1}{2}$, which implies that X is of bounded variation, whence X is Λ-strongly convergent. It seems to be unresolved if there is a sequence X which is Λ^2-strongly convergent but not Λ-strongly convergent.

2. New results

The above observations show that we need to define Λ^2-strong convergence in a different way as in [1]. Here we give an appropriate definition. Let $\Lambda = \{\lambda_k\}$ be a nondecreasing sequence of positive numbers tending to ∞. A sequence $X = \{x_k\}$ of complex numbers converges Λ^2-strongly to a complex number x if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} \big| \lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x) \big| = 0$$

with the agreement $\lambda_{-1} = \lambda_{-2} = x_{-1} = x_{-2} = 0$. It is easy to see that if X converges Λ-strongly, then it converges Λ^2-strongly, it is enough to consider

$$\sum_{k=0}^{n} \big| \lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x) \big| \leqq 2 \sum_{k=0}^{n} \big| \lambda_k(x_k - x) - \lambda_{k-1}(x_{k-1} - x) \big| .$$

Moreover, if X converges Λ^2-strongly, then it converges in the ordinary sense since

$$x_k - x = \frac{1}{\lambda_n} \sum_{\substack{0 \leqq k \leqq n \\ 2 | n-k}} \big(\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x) \big) .$$

Thus, Λ^2-strong convergence is an intermediate notion between Λ-strong convergence and ordinary convergence. We also give an example for a Λ^2-strongly convergent but not Λ-strongly convergent sequence.

EXAMPLE. Let $x_k = (-1)^{k+1}\frac{1}{k+1}$ and $\lambda_k = k+1$. Then $x = \lim_k x_k = 0$ and

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\left|(k+1)(-1)^{k+1}\frac{1}{k+1} - (k-1)(-1)^{k-1}\frac{1}{k-1}\right| = \lim_{n\to\infty} 0 = 0,$$

but

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=1}^{n}\left|(k+1)(-1)^{k+1}\frac{1}{k+1} - k(-1)^{k}\frac{1}{k}\right| = \lim_{n\to\infty}\frac{2n}{n+1} = 2.$$

We formulate two results analogous to Lemma M1 and M2.

LEMMA 1. *A sequence X converges Λ^2-strongly to a number x if and only if condition* (i) *is satisfied and*

$$\text{(II)} \qquad \lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k=2}^{n}\lambda_{k-2}|x_k - x_{k-2}| = 0.$$

PROOF. The representation

$$\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x) = (\lambda_k - \lambda_{k-2})(x_k - x) + \lambda_{k-2}(x_k - x_{k-2})$$

implies both

$$\frac{1}{\lambda_n}\sum_{k=0}^{n}\left|\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)\right|$$

$$\leqq \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-2})|x_k - x| + \frac{1}{\lambda_n}\sum_{k=2}^{n}\lambda_{k-2}|x_k - x_{k-2}|$$

and

$$\frac{1}{\lambda_n}\sum_{k=2}^{n}\lambda_{k-2}|x_k - x_{k-2}|$$

$$\leqq \frac{1}{\lambda_n}\sum_{k=0}^{n}\left|\lambda_k(x_k - x) - \lambda_{k-2}(x_{k-2} - x)\right| + \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-2})|x_k - x|.$$

Using the above inequalities together with the fact that for any x_k converging to x it is known that

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-2})|x_k - x| = 0,$$

we get the necessity and the sufficiency of the two conditions (i) and (II). □

Lemma 2. *A sequence X converges Λ^2-strongly to a number x if and only if*

$$\sigma_n := \frac{1}{\lambda_n} \sum_{\substack{0 \leqq k \leqq n \\ 2 | n-k}} (\lambda_k - \lambda_{k-2}) x_k$$

converges to x in the ordinary sense and condition (II) *is satisfied.*

Proof. Clearly,

$$x_n - \sigma_n = \frac{1}{\lambda_n} \sum_{\substack{0 \leqq k \leqq n \\ 2 | n-k}} (\lambda_k - \lambda_{k-2})(x_n - x_k)$$

$$= \frac{1}{\lambda_n} \sum_{\substack{0 \leqq k \leqq n \\ 2 | n-k}} (\lambda_k - \lambda_{k-2}) \sum_{\substack{k+2 \leqq j \leqq n \\ 2 | n-j}} (x_j - x_{j-2})$$

$$= \frac{1}{\lambda_n} \sum_{\substack{2 \leqq j \leqq n \\ 2 | n-j}} (x_j - x_{j-2}) \sum_{\substack{0 \leqq k \leqq j-2 \\ 2 | n-k}} (\lambda_k - \lambda_{k-2}) = \frac{1}{\lambda_n} \sum_{\substack{2 \leqq j \leqq n \\ 2 | n-j}} \lambda_{j-2}(x_j - x_{j-2}).$$

Hence

$$\limsup_{n\to\infty} |x_n - \sigma_n| \leqq \limsup_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=2}^{n} \lambda_{k-2} |x_k - x_{k-2}|.$$

According to Lemma 1, for the necessity part, it is enough to see the that $\lim_n \sigma_n = x$, which comes from the above inequality, (II) and $\lim_n x_n = x$. For the sufficiency part, we only need $\lim_n x_n = x$, which comes from the above inequality, (II) and $\lim_n \sigma_n = x$. □

We remark that we can also define Λ^r-strong convergence for an arbitrary integer $r \geqq 3$. We can say, that for a $\Lambda = \{\lambda_k\}$ nondecreasing sequence of positive numbers tending to ∞, a sequence $X = \{x_k\}$ of complex numbers converges Λ^r-strongly to a complex number x if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} \big| \lambda_k (x_k - x) - \lambda_{k-r}(x_{k-r} - x) \big| = 0$$

with the agreement $\lambda_{-1} = \ldots = \lambda_{-r} = x_{-1} = \ldots = x_{-r} = 0$. One can easily show that these convergence notions are also intermediate notions between

Λ-strong convergence and ordinary convergence. Moreover, the following two analogous results can be shown in a similar way as above.

PROPOSITION 1. *A sequence X converges Λ^r-strongly to a number x if and only if condition* (i) *is satisfied and*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=r}^{n} \lambda_{k-r} |x_k - x_{k-r}| = 0. \tag{II$'$}$$

PROPOSITION 2. *A sequence X converges Λ^r-strongly to a number x if and only if*

$$\sigma_n := \frac{1}{\lambda_n} \sum_{\substack{0 \leqq k \leqq n \\ r \mid n-k}} (\lambda_k - \lambda_{k-r}) x_k$$

converges to x in the ordinary sense and condition (II$'$) *is satisfied.*

Acknowledgement. The author thanks Professor Ferenc Móricz for his valuable comments and suggestions regarding this paper.

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