Research and Reports on Mathematics

Short Communication

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Geometric Proof of the Sum of Geometric Series

Korus P*

Introduction

The well-known formula for the sum of the geometric series is

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

For arbitrary -1<q<1. Among analytic proofs, geometric proofs were also given for this formula, see [1], mostly for 0<q<1. Now we prove that for any 0<q<1

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

holds in the 'Positive case' and

$$s' = \sum_{k=1}^{\infty} (-1)^{k-1} q^k = \frac{q}{1-q}$$

in the 'Alternating case'.

Positive case

As in Figure 1, we do the following process.

Step 1: Take a unit square S_1 and take rectangles Q_1 with area A_{Q_1} = q and Q_2 with A_{Q_2} = q_2. We also

take 'adjunct' rectangles R_1 with $A_{R_1} = \frac{1-2q}{q} A_{Q1}$ and R_2 with $A_{R_2} = \frac{1-2q}{q} A_{Q_2}$.

We get a remaining square S_2 of side length q.

Step 2: We repeat the actions of the previous step for S_2, but we reduce the rectangles by a scale factor of q. Then we get Q_3; Q_4; R_3; R_4 with $A_{Q_4} = q^4$, $A_{Q_4} = q^4$, $A_{R_3} = \frac{1-2q}{q} A_{Q_3}$, $A_{R_3} = \frac{1-2q}{q} A_{Q_3}$ and a remaining square S_3 of side length q^2.

General step *k*: We repeat the actions of the previous step for S_k, but we reduce the rectangles by a scale factor of q.

We get that the area of unit square S_1 is

$$1 = \sum_{k=1}^{\infty} A_{Q_k} + \sum_{k=1}^{\infty} A_{R_k} = s + \frac{1-2q}{q} S = \frac{1-q}{q} s,$$

Hence $S = \dfrac{q}{1-q}$

Alternating case

As in Figure 2, we do the following process.

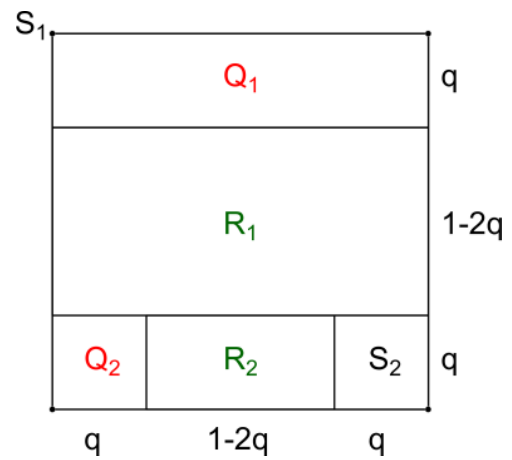

Figure 1: Positive case.

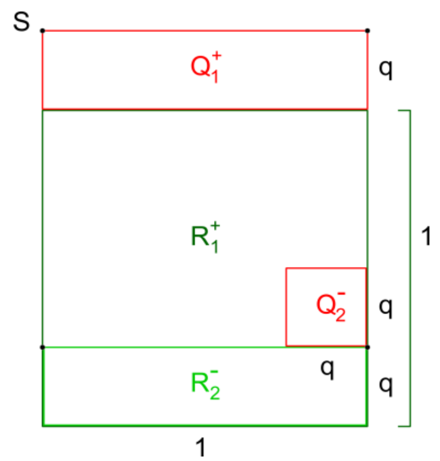

Figure 2: Alternating case.

Step 1: Take a unit square S and take rectangles Q_1^+ with $A_{Q_1^+} = q$ and Q_2^- with $A_{Q_2^-} = q^2$. We also take 'adjunct' square R_1^+ of side length 1 with $A_{R_1^+} = \frac{1}{q} A_{Q_1^+}$ and rectangle R_2^- of side lengths q and 1 with $A_{R_2^-} = \frac{1}{q} A_{Q_1^+}$. (Plus and minus signs indicate the signs of the areas of rectangles within the final sum.)

Step 2: We repeat the actions of the previous step for square Q_2^- of side length q, but we reduce the rectangles by a scale factor of q. Then we get $Q_3^+, R_3^+, R_3^+, R_4^-$ with $A_{Q_4^-} = q^4$, $A_{Q_4^-} = q^4$, $A_{R_3^+} = \frac{1}{q} A_{Q_3^+}$, $A_{R_4^-} = \frac{1}{q} A_{Q_4^-}$.

General step *k*: We repeat the actions of the previous step for Q_{2k-2}^-, but we reduce the rectangles by a scale factor of q.

***Corresponding author:** Korus P, Department of Mathematics, Juhasz Gyula Faculty of Education University of Szeged, Hattyas utca 10, H-6725 Szeged, Hungary, E-mail: korpet@jgypk.u-szeged.hu

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We get that the area of unit square S_1' is

$$1=\sum_{k=1}^{\infty}\left(A_{Q_{2k-1}^{+}}-A_{Q_{2k}^{-}}+A_{R_{2k-1}^{+}}-A_{R_{2k}^{-}}\right)=\sum_{k=1}^{\infty}(-1)^{k-1}q^{k}+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{q}q^{k}$$
$$=s+\frac{1}{q}s=\frac{1+q}{q}s$$

Hence $s'=\dfrac{q}{1+q}$.

References

1. Nelsen RB (1993) Proofs without words: Exercises in visual thinking. MAA.

Author Affiliation Top

Department of Mathematics, Juhasz Gyula Faculty of Education University of Szeged, Hungary