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Short Communication

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Geometric Proof of the Sum of Geometric Series

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Introduction

The well-known formula for the sum of the geometric series is

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

For arbitrary -1 < q < 1. Among analytic proofs, geometric proofs were also given for this formula, see [1], mostly for 0 < q < 1. Now we prove that for any 0 < q < 1

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

holds in the 'Positive case' and

$$s' = \sum_{k=1}^{\infty} (-1)^{k-1} q^k = \frac{q}{1-q}$$

in the 'Alternating case'.

Positive case

As in Figure 1, we do the following process.

Step 1: Take a unit square S_i and take rectangles Q_i with area $A_{Q_1} = q$ and Q_2 with $A_{Q_2} = q_2$. We also

take 'adjunct' rectangles R_1 with $A_{R_1} = \frac{1-2q}{q} A_{Q1}$ and R_2 with $A_{R_2} = \frac{1-2q}{q} A_{Q2}$.

We get a remaining square S_2 of side length q.

Step 2: We repeat the actions of the previous step for S_2 , but we reduce the rectangles by a scale factor of q. Then we get Q_3 ; Q_4 ; R_3 ; R_4 with $A_{Q_4} = q^4$, $A_{Q_4} = q^4$, $A_{R_3} = \frac{1-2q}{q}A_{Q_3}$, $A_{R_5} = \frac{1-2q}{q}A_{Q_5}$ and a remaining square S_3 of side length q^2 .

General step *k*: We repeat the actions of the previous step for S_{k^2} but we reduce the rectangles by a scale factor of *q*.

We get that the area of unit square S_1 is

$$1 = \sum_{k=1}^{\infty} A_{Q_k} + \sum_{k=1}^{\infty} A_{R_k} = s + \frac{1-2q}{q} S = \frac{1-q}{q} s,$$

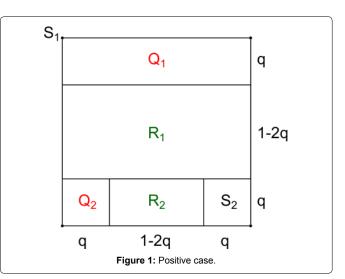
Hence $S = \frac{q}{1-q}$
Alternating case

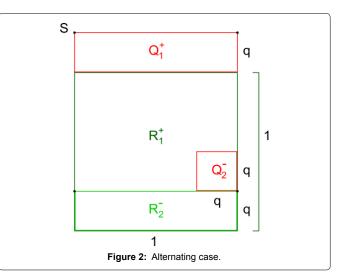
As in Figure 2, we do the following process.

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Step 1: Take a unit square *S* and take rectangles Q_1^+ with $A_{Q_1^+} = q$ and Q_2^- with $A_{Q_2^-} = q^2$. We also take 'adjunct' square R_1^+ of side length 1 with $A_{R_1^+} = \frac{1}{q} A_{Q_1^+}$ and rectangle R_2^- of side lengths q and 1 with $A_{R_2^-} = \frac{1}{q} A_{Q_1^+}$. (Plus and minus signs indicate the signs of the areas of rectangles within the final sum.)

Step 2: We repeat the actions of the previous step for square Q_2^- of side length q, but we reduce the rectangles by a scale factor of q. Then we get Q^+, P^+, P^- with $A_{QP} = q^4, A_{QP} = q^4, A_{PP} = \frac{1}{2}A_{PP}$.

$$\begin{array}{l} \text{Inen we get } Q_{1}^{+}, R_{3}^{+}, R_{3}^{+}, R_{4}^{-} \text{ with } A_{\underline{Q}_{4}^{-}} - q \ , \ A_{\underline{Q}_{4}^{-}} - q \ , \ A_{\underline{R}_{5}^{+}} = -A_{\underline{Q}_{4}} \\ A_{\underline{R}_{4}^{-}} = \frac{1}{q} A_{\underline{Q}_{4}} \end{array}$$

General step *k*: We repeat the actions of the previous step for Q_{2k-2}^- , but we reduce the rectangles by a scale factor of *q*.

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We get that the area of unit square $S_1^{'}$ is

$$\begin{split} &1 = \sum_{k=1}^{\infty} \left(A_{Q_{2k-1}^{+}} - A_{Q_{2k}^{-}} + A_{R_{2k-1}^{+}} - A_{R_{2k}^{-}} \right) = \sum_{k=1}^{\infty} (-1)^{k-1} q^{k} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{q} q^{k} \\ &= s + \frac{1}{q} s = \frac{1+q}{q} s \end{split}$$

Hence
$$s' = \frac{q}{1+q}$$
.

References

1. Nelsen RB (1993) Proofs without words: Exercises in visual thinking. MAA.

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