THE ASYMPTOTIC NUMBER OF PLANAR, SLIM, SEMIMODULAR LATTICE DIAGRAMS

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ABSTRACT. A lattice L is *slim* if it is finite and the set of its join-irreducible elements contains no three-element antichain. We prove that there exists a positive constant C such that, up to similarity, the number of planar diagrams of slim semimodular lattices of size n is asymptotically $C \cdot 2^n$.

1. Introduction and the result

A finite lattice L is *slim* if $\mathrm{Ji}\, L$, the set of join-irreducible elements of L, contains no three-element antichain. Equivalently, L is slim if $\mathrm{Ji}\, L$ is the union of two chains. Slim, semimodular lattices were heavily used while proving a recent generalization of the classical Jordan-Hölder theorem for groups in [4]. These lattices are *planar*, that is, they have planar diagrams, see [4]. Hence it is reasonable to study their planar diagrams, which are called *slim, semimodular* (*lattice*) *diagrams* for short. The *size* of a diagram is the number of elements of the lattice it represents. Let D_1 and D_2 be two planar lattice diagrams. A bijection $\phi\colon D_1 \to D_2$ is a *similarity map* if it is a lattice isomorphism preserving the left-right order of (upper) covers and that of lower covers of each element of D_1. If there is a similarity map $D_1 \to D_2$, then these two diagrams are *similar*, and we will treat them as equal. Let $N_{\mathrm{ssd}}(n)$ denote the number of slim, semimodular diagrams of size n, counting them up to similarity. Our target is to prove the following result.

Theorem 1.1. *There exists a positive constant $C < 1$ such that $N_{\mathrm{ssd}}(n)$ is asymptotically $C \cdot 2^n$, that is,* $\lim_{n\to\infty}\big(N_{\mathrm{ssd}}(n)/2^n\big) = C$.

Given two composition series in a finite group, the intersections of their members form a slim semimodular lattice with respect to "$\supseteq$". This follows from Wielandt [13]; see also the proof of Nation [11, Theorem 9.8]. Conversely, [6] proves that every slim semimodular lattice can be represented in this way. Therefore, in a reasonable, order theoretic sense, Theorem 1.1 tells us how many ways the members of two composition series in a group can intersect each other, provided that there are exactly n intersections and that we make a distinction between the first composition series and the second one.

Note that there are two different methods to deal with $N_{\mathrm{ssd}}(n)$. The present one yields the asymptotic statement above, while the method of [1] gives the exact

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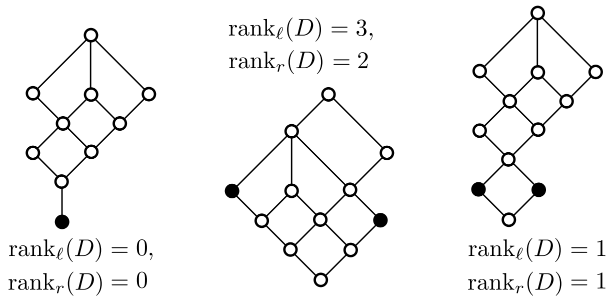

Figure 1. Left and right ranks

values of $N_{\text{ssd}}(n)$ up to $n = 50$ (with the help of a usual personal computer). Also, [1] determines the number $N_{\text{ssl}}(n)$ of slim, semimodular *lattices* of size n up to $n = 50$ while we do not even know $\lim_{n\to\infty}\big(N_{\text{ssl}}(n)/N_{\text{ssl}}(n-1)\big)$, and it is only a conjecture that this limit exists.

Note also that, besides [1] and [2], there are several papers on counting lattices; see, for example, M. Erné, J. Heitzig, and J. Reinhold [7], M. M. Pawar and B. N. Waphare [12], and J. Heitzig and J. Reinhold [9].

2. Lattice theoretic lemmas

A minimal non-chain region of a planar lattice diagram D is called a *cell*. A four-element cell is a 4-*cell*; it is also a *covering square*, that is, a cover-preserving four-element Boolean sublattice. We say that D is a 4-*cell diagram* if all of its cells are 4-cells. We shall heavily rely on the following result of G. Grätzer and E. Knapp [8, Lemmas 4 and 5].

Lemma 2.1. *Let D be a finite planar lattice diagram.*

- (i) *If D is semimodular, then it is a* 4*-cell diagram. If A and B are* 4*-cells of D with the same bottom, then these* 4*-cells have the same top.*
- (ii) *If D is a* 4*-cell diagram in which no two* 4*-cells with the same bottom have distinct tops, then D is semimodular.*

In what follows, we always assume that $4 \le n \in \mathbb{N} = \{1, 2, \dots\}$, and that D is a slim, semimodular diagram of size n. Let w_D^ℓ be the smallest doubly irreducible element of the left boundary chain $\mathrm{BC}_\ell(D)$ of D, and let $\mathrm{rank}_\ell(D)$ be the height of w_D^ℓ. The left-right duals of these concepts are denoted by w_D^r and $\mathrm{rank}_r(D)$. See Figure 1 for an illustration, where w_D^ℓ and w_D^r are the black-filled elements. By D. Kelly and I. Rival [10, Proposition 2.2], each planar lattice diagram with at least three elements contains a doubly irreducible element $\neq 0, 1$ on its left boundary. This implies the following statement, on which we will rely implicitly.

Lemma 2.2. *Either* $\mathrm{rank}_\ell(D) = \mathrm{rank}_r(D) = 0$ *and* $w_D^\ell = w_D^r = 0$, *or* $\mathrm{rank}_\ell(D) > 0$ *and* $\mathrm{rank}_r(D) > 0$.

For $a \in D$, the ideal $\{x \in D : x \le a\}$ is denoted by $\downarrow a$.

Lemma 2.3. $\mathrm{BC}_\ell(D) \cap \downarrow w_D^\ell \subseteq \mathrm{Ji}\, D$.

Proof. Suppose, for a contradiction, that the lemma fails, and let u be the smallest join-reducible element belonging to $\mathrm{BC}_\ell(D) \cap \downarrow w_D^\ell$. By D. Kelly and I. Rival [10, Proposition 2.2], there is a doubly irreducible element v of the ideal $\downarrow u = \{x \in D : x \le u\}$ such that $v \in \mathrm{BC}_\ell(\downarrow u)$; notice that v also belongs to $\mathrm{BC}_\ell(D)$. Clearly, $v < u$ and v is join-irreducible in D. Therefore, since $v < u \le w_D^\ell$ and w_D^ℓ is the least doubly irreducible element of $\mathrm{BC}_\ell(D)$, v is meet-reducible in D. Hence there exist a $p \in D$ such that $v \prec p$ and $p \notin \downarrow u$. Denote by u_0 the unique lower cover of u in $\mathrm{BC}_\ell(D)$. Since $v < u$, we have that $v \le u_0$. By semimodularity and $p \not\le u_0$, we obtain that $u_0 = u_0 \vee v \prec u_0 \vee p \ne u$. Hence u_0 has two covers, u and $u_0 \vee p$. Thus $u_0, u \in \mathrm{BC}_\ell(D)$, $u_0 \prec u$, u is join-reducible, and u_0 is meet-reducible. This contradicts [5, Lemma 4]. ☐

Next, we prove the following lemma.

Lemma 2.4. *For* $4 \le n \in \mathbb{N}$*, we have that*

$$N_{\mathrm{ssd}}(n-1) + N_{\mathrm{ssd}}(n-3) \le N_{\mathrm{ssd}}(n), \tag{2.1}$$

$$N_{\mathrm{ssd}}(n) \le 2 \cdot N_{\mathrm{ssd}}(n-1). \tag{2.2}$$

Proof. The set of slim, semimodular diagrams of size n is denoted by $\mathrm{SSD}(n)$. Let

$$\begin{aligned} \mathrm{SSD}_{00}(n) &= \{D \in \mathrm{SSD}(n) : \mathrm{rank}_\ell(D) = \mathrm{rank}_r(D) = 0\}, \\ \mathrm{SSD}_{11}(n) &= \{D \in \mathrm{SSD}(n) : \mathrm{rank}_\ell(D) = \mathrm{rank}_r(D) = 1\}, \text{ and} \\ \mathrm{SSD}_{++}(n) &= \mathrm{SSD}(n) - \mathrm{SSD}_{00}(n). \end{aligned}$$

Since we can omit the least element and the least three elements, respectively, and the remaining diagram is still slim and semimodular by Lemma 2.1, we conclude that $|\mathrm{SSD}_{00}(n)| = N_{\mathrm{ssd}}(n-1)$ and $|\mathrm{SSD}_{11}(n)| = N_{\mathrm{ssd}}(n-3)$. This implies (2.1). For $D \in \mathrm{SSD}_{++}(n)$, we define

$$D^* = D - \{w_D^\ell\}.$$

We know from By D. Kelly and I. Rival [10, Proposition 2.2], mentioned earlier, that

$$w_D^\ell \notin \{0, 1\}, \text{ provided } D \in \mathrm{SSD}_{++}(n). \tag{2.3}$$

This, together with the fact that $D \in \mathrm{SSD}_{++}(n)$ is not a chain, yields that

$$\mathrm{length}\, D^* = \mathrm{length}\, D. \tag{2.4}$$

Let ${w_D^\ell}^-$ denote the unique lower cover of w_D^ℓ in D. Since each meet-reducible element has exactly two covers by [5, Lemma 2], we conclude from Lemma 2.3 that

$$w_{D^*}^\ell = {w_D^\ell}^-. \tag{2.5}$$

It follows from Lemma 2.1 that $D^* \in \mathrm{SSD}(n-1)$. From (2.5) we obtain that

$$D^* \in \mathrm{SSD}(n-1) \text{ determines } D. \tag{2.6}$$

Hence $|\mathrm{SSD}_{++}(n)| \le |\mathrm{SSD}(n-1)| = N_{\mathrm{ssd}}(n-1)$. Combining this with $|\mathrm{SSD}_{00}(n)| = N_{\mathrm{ssd}}(n-1)$ and $\mathrm{SSD}(n) = \mathrm{SSD}_{00}(n) \,\dot\cup\, \mathrm{SSD}_{++}(n)$, where $\dot\cup$ stands for disjoint union, we obtain (2.2). ☐

Next, let

$$W(n) = \mathrm{SSD}(n-1) - \{D^* : D \in \mathrm{SSD}_{++}(n)\}. \tag{2.7}$$

Fortunately, this set is relatively small by the following lemma. The upper integer part of a real number r is denoted by $\lceil x \rceil$; for example, $\lceil \sqrt{3}\, \rceil = 2$.

Lemma 2.5. *If* $4 \le n$*, then* $|W(n)| \le \displaystyle\sum_{j=2}^{n+1-\lceil\sqrt{n-1}\,\rceil} N_{\mathrm{ssd}}(j)$.

Proof. First we show that

$$W(n) = \{E \in \mathrm{SSD}(n-1) : w_E^\ell \text{ is a coatom of } E\}. \tag{2.8}$$

The $\supseteq$ inclusion is clear from (2.3), (2.4), and (2.5). These facts together with Lemma 2.1 also imply the reverse inclusion since by adding a new cover to w_E^ℓ, to be positioned to the left of $\mathrm{BC}_\ell(E)$, we obtain a slim semimodular diagram D such that $D^* = E$.

It follows from Lemma 2.3 that no down-going chain starting at w_E^ℓ can branch out. Thus

$$\downarrow w_E^\ell \subseteq \mathrm{BC}_\ell(E) \text{ and } \downarrow w_E^\ell \text{ is a chain.} \tag{2.9}$$

Since w_E^ℓ is a coatom, we have that

$$\text{with the notation } E^\blacktriangleleft = E \setminus \downarrow w_E^\ell, \quad |E^\blacktriangleleft| = |E| - \mathrm{length}\, E. \tag{2.10}$$

Clearly, $E^\blacktriangleleft$ is a join-subsemilattice of E since it is an order-filter. To prove that

$$E^\blacktriangleleft \text{ is a slim, semimodular diagram,} \tag{2.11}$$

assume that $x, y \in E^\blacktriangleleft - \{1\}$. We want to show that $x \wedge y$, taken in E, belongs to $E^\blacktriangleleft$. Let x_0 and y_0 be the smallest element of $\mathrm{BC}_\ell(E) \cap \downarrow x$ and $\mathrm{BC}_\ell(E) \cap \downarrow y$, respectively. Since $x_0, y_0 \in \mathrm{BC}_\ell(E) \cap \big(\downarrow w_E^\ell - \{w_E^\ell\}\big)$, the definition of w_E^ℓ implies that x_0 and y_0 are meet-reducible. Hence they have exactly two covers by [5, Lemma 2]. Let x_1 and y_1 denote the cover of x_0 and y_0, respectively, that do not belong to $\mathrm{BC}_\ell(E)$, and let x^+ and y^+ be the respective covers belonging to $\mathrm{BC}_\ell(E)$. By the choice of x_0, we have that $x^+ \not\le x$, whence $x_1 \le x$. Similarly, $y_1 \le y$. Since $\mathrm{BC}_\ell(E)$ is a chain and the case $x_0 = y_0$ will turn out to be trivial, we can assume that $x_0 < y_0$. We know that $x_1 \not\le y_0$ since otherwise x_1 would belong to $\mathrm{BC}_\ell(E)$ by (2.9). Using semimodularity, we obtain that $x_1 \vee y_0 \succ y_0$. Since y_0 has only two covers by [5, Lemma 2] and $x_1 \le y^+$ would imply $x_1 \in \mathrm{BC}_\ell(E)$ by (2.9), it follows that $x_1 \vee y_0 = y_1$. Hence $x_1 \le y$, $x_1 \le x$, and $x_1 \in E^\blacktriangleleft$ imply that $x \wedge y$ belongs to (the order filter) $E^\blacktriangleleft$. Thus $E^\blacktriangleleft$ is (to be more precise, determines) a sublattice of (the lattice determined by) E. The semimodularity of $E^\blacktriangleleft$ follows from Lemma 2.1. This proves (2.11).

By (2.10) and (2.11), a trivial argument gives that

$$E^\blacktriangleleft \in \mathrm{SSD}(n - \mathrm{length}\, E) \text{ and } E^\blacktriangleleft \text{ determines } E. \tag{2.12}$$

Next, we have to determine what values $h = \mathrm{length}\, E$ can take. Clearly, $h \le |E| - 1 = n - 2$. There are various ways to check that $|E| \le (1 + \mathrm{length}\, E)^2 = (1+h)^2$; this follows from the main theorem of [6], and follows also from the proof of [3, Corollary 2]. Since now $|E| = n - 1$, we obtain that $\lceil\sqrt{n-1}\,\rceil - 1 \le h$. Therefore, combining (2.11) and (2.12), we obtain that

$$W(n) \le \sum_{h=\lceil\sqrt{n-1}\,\rceil-1}^{n-2} N_{\mathrm{ssd}}(n-h).$$

Substituting j for $n - h$, we obtain our statement. $\square$

We conclude this section by the following lemma.

Lemma 2.6. $2 \cdot N_{\rm ssd}(n-1) - \sum_{j=2}^{n+1-\lceil\sqrt{n-1}\,\rceil} N_{\rm ssd}(j) \le N_{\rm ssd}(n) \le 2 \cdot N_{\rm ssd}(n-1).$

Proof. By (2.6) and the definition of $W(n)$, we have that

$$\begin{aligned} N_{\rm ssd}(n) &= |\mathrm{SSD}_{00}(n)| + |\mathrm{SSD}_{++}(n)| = N_{\rm ssd}(n-1) + |\mathrm{SSD}(n-1) - W(n)| \\ &= N_{\rm ssd}(n-1) + N_{\rm ssd}(n-1) - |W(n)|, \end{aligned}$$

and the statement follows from Lemma 2.5 and (2.2). □

3. Tools from Analysis at work

For $k \ge 2$, define $\kappa_k = N_{\rm ssd}(k)/N_{\rm ssd}(k-1)$. Since $N_{\rm ssd}(n-3)/N_{\rm ssd}(n-1) = 1/(\kappa_{n-1}\kappa_{n-2})$, dividing the inequalities of Lemma 2.4 by $N_{\rm ssd}(n-1)$ we obtain that $1 + 1/(\kappa_{n-1}\kappa_{n-2}) \le \kappa_n \le 2$, for $n \ge 4$. Furthermore, in view of the sentence following (2.7), (2.8) implies easily that $\kappa_n < 2$ if $n \ge 7$. Therefore, since $\kappa_k \le 2$ also holds for $k \in \{2,3\}$ and $1 + 1/(2\cdot 2) = 5/4$, we conclude that

$$5/4 \le \kappa_n \le 2, \quad \text{for} \quad n \ge 4, \quad \text{and} \quad \kappa_n < 2, \quad \text{for} \quad n \ge 7. \tag{3.1}$$

Clearly, $N_{\rm ssd}(k-1) = N_{\rm ssd}(k)/\kappa_k \le \frac{4}{5} \cdot N_{\rm ssd}(k)$ if $k \ge 4$. Thus, by iteration, we obtain that

$$N_{\rm ssd}(k-j) \le (4/5)^j \cdot N_{\rm ssd}(k), \qquad \text{for } j \in \mathbb{N}_0 \text{ and } k \ge j+4. \tag{3.2}$$

If $k \ge 5$, then using $N_{\rm ssd}(k) \ge N_{\rm ssd}(5) \ge 3$ (actually, $N_{\rm ssd}(5) = 3$), we obtain that

$$\begin{aligned} N_{\rm ssd}(1) + \dots + N_{\rm ssd}(k) &= 1 + 1 + 1 + N_{\rm ssd}(4) + \dots + N_{\rm ssd}(k) \\ &\le 3 + N_{\rm ssd}(k) \cdot \big((4/5)^{k-4} + (4/5)^{k-5} + \dots + (4/5)^0\big) \\ &\le N_{\rm ssd}(k) + N_{\rm ssd}(k) \cdot 1/(1-4/5) = 6N_{\rm ssd}(k). \end{aligned} \tag{3.3}$$

Combining Lemma 2.6 with (3.3) and (3.2), we obtain that

$$\begin{aligned} &2N_{\rm ssd}(n-1) - 6 \cdot (4/5)^{\lceil\sqrt{n-1}\,\rceil - 2} \cdot N_{\rm ssd}(n-1) \le \\ &\qquad 2N_{\rm ssd}(n-1) - 6N_{\rm ssd}(n+1-\lceil\sqrt{n-1}\,\rceil) \\ &\qquad \le N_{\rm ssd}(n) \le 2N_{\rm ssd}(n-1). \end{aligned}$$

Dividing the formula above by $2N_{\rm ssd}(n-1)$ and (3.1) by 2, we obtain that

$$\max\big(5/8\,,\, 1 - 3 \cdot (4/5)^{\lceil\sqrt{n-1}\,\rceil - 2}\big) \le \kappa_n/2 \le 1, \quad \text{for } n \ge 5. \tag{3.4}$$

Next, let us choose an integer $m \ge 5$, and define

$$z_0 = z_0(m) = \min\big(3/8\,,\, 3 \cdot (4/5)^{\lceil\sqrt{m-1}\,\rceil - 2}\big).$$

Lemma 3.1. *For $0 \le z < 1$, we have $-\ln(1-z) \le z/(1-z)$. If, in addition, $0 \le z \le z_0$, then $z/(1-z) \le z/(1-z_0)$.*

Proof. The second inequality is obvious. The first inequality holds for $z = 0$ and, for $0 \le z < 1$, the derivative $1/(1-z)$ of the left side is less than $1/(1-z)^2$, that of the right side. This implies the first inequality. □

With the auxiliary steps made so far, we are ready to start the final argument.

Proof of Theorem 1.1. For $n > m$, let

$$p_n = \prod_{j=m+1}^{n} (\kappa_j/2).$$

We obtain from (3.4) that $\{p_n\}$, that is, $\{p_n\}_{n=m+1}^{\infty}$, is a decreasing sequence of positive numbers. Clearly,

$$N_{\mathrm{ssd}}(n)/2^n = p_n \cdot N_{\mathrm{ssd}}(m)/2^m. \tag{3.5}$$

Hence it suffices to prove that the sequence $\{p_n\}$ converges to a positive number, because then its limit is smaller than 1 by (3.1). Let $s_n = -\ln p_n$, $\mu = 3(1-z_0)^{-1}$, $\alpha = 4/5$, and $\nu = 5\mu/4 = \mu/\alpha$. Note that $\{s_n\}$ is an increasing sequence.

Using (3.4) together with Lemma 3.1 at the inequality marked with $\le'$ below and (3.4) at the one marked with $\le^*$, and using that the function $f(x) = \alpha^{\sqrt{x}}$ is decreasing, we obtain that

$$\begin{aligned} 0 < s_n &= \sum_{j=m+1}^{n} \bigl(-\ln(\kappa_j/2)\bigr) \le' \sum_{j=m+1}^{n} (1-\kappa_j/2)/(1-z_0) \\ &\le^* \mu \cdot \sum_{j=m+1}^{n} \alpha^{\lceil \sqrt{j-1} \rceil - 2} \le \mu \cdot \sum_{j=m+1}^{n} \alpha^{\sqrt{j-1}-1} = \mu \cdot \sum_{k=m}^{n-1} \alpha^{\sqrt{k}-1} \\ &= \nu \cdot \sum_{k=m}^{n-1} \alpha^{\sqrt{k}} \le \nu \cdot \int_{x=m-1}^{n-1} \alpha^{\sqrt{x}} dx \le \nu \cdot \bigl(F(\infty) - F(m-1)\bigr), \end{aligned}$$

where $F(x)$ is a function whose derivative is $f(x)$. Let $\delta = -\ln \alpha = \ln(5/4)$. It is routine to check (by hand or by computer algebra) that, up to a constant summand,

$$F(x) = -2 \cdot \delta^{-2} \cdot (1 + \delta\sqrt{x}) \cdot \alpha^{\sqrt{x}}.$$

Clearly, $F(\infty) = \lim_{x\to\infty} F(x) = 0$. This proves that the sequence $\{s_n\}$ converges; so does $\{p_n\} = \{e^{-s_n}\}$ by the continuity of the exponential function. Therefore, since $N_{\mathrm{ssd}}(m)/2^m$ in (3.5) does not depend on n, we conclude Theorem 1.1. □

Remark 3.2. We can approximate the constant in Theorem 1.1 as follows. Since $e^{-\nu\cdot(F(\infty)-F(m))} \le e^{-s_n} = p_n \le 1$ and, by (3.5), $C = \lim_{n\to\infty}\bigl(p_n N_{\mathrm{ssd}}(m)/2^m\bigr)$, we obtain that

$$e^{\nu F(m)} \cdot N_{\mathrm{ssd}}(m)/2^m = e^{-\nu\cdot(F(\infty)-F(m))} \cdot N_{\mathrm{ssd}}(m)/2^m \le C \le N_{\mathrm{ssd}}(m)/2^m. \tag{3.6}$$

Unfortunately, our computing power yields only a very rough estimation. The largest m such that $N_{\mathrm{ssd}}(50)$ is known is $m = 50$, see [1]. With $m = 50$ and $N_{\mathrm{ssd}}(m) = N_{\mathrm{ssd}}(50) = 81\,287\,566\,224\,125$, it is a routine task to turn (3.6) into

$$0.42 \cdot 10^{-57} \le C \le 0.073\ .$$

We have reasons (but no proof) to believe that $0.023 \le C \le 0.073$, see the Maple worksheet (version V) available from the authors's home page.

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