PLANAR GRADED LATTICES AND THE $\mathbf{c_1}$-MEDIAN PROPERTY

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Abstract. Let L be a lattice of finite length, $\xi = (x_1, \ldots, x_k) \in L^k$, and $y \in L$. The *remoteness* $r(y, \xi)$ of y from ξ is $d(y, x_1) + \cdots + d(y, x_k)$, where d stands for the minimum path length distance in the covering graph of L. Assume, in addition, that L is a graded planar lattice. We prove that whenever $r(y, \xi) \leq r(z, \xi)$ for all $z \in L$, then $y \leq x_1 \vee \cdots \vee x_k$. In other words, L satisfies the so-called c_1-*median property.*

1. Introduction

Let L be a lattice of finite length, $\xi = (x_1, \ldots, x_k) \in L^k$, and $y \in L$. The *remoteness* $r(y, \xi)$ of y from ξ is $d(y, x_1) + \cdots + d(y, x_k)$, where d stands for the minimum path length distance in the covering graph of L. The set of *medians* of ξ is $M(\xi) = \{y \in L : r(y, \xi) \leq r(z, \xi)$ for all $z \in L\}$. The determination of median sets based on different types of metric spaces is an important problem in mathematics with applications in areas such as cluster analysis and social choice [2], consensus and location [4] [9], and classification theory [1].

The determination of median sets in terms of the ordering on L leads to some interesting results. For any $\xi = (x_1, \ldots, x_k) \in L^k$ and for any integer t such that $1 \leq t \leq k$ we let

$$c_t(\xi) = \bigvee \{ \bigwedge_{i \in I} x_i : I \subseteq \{1, \ldots, k\}, |I| = t\}$$

and

$$c'_t(\xi) = \bigwedge \{ \bigvee_{i \in I} x_i : I \subseteq \{1, \ldots, k\}, |I| = t\}.$$

In 1980, Monjardet [10] showed that if L is a finite distributive lattice, then

$$M(\xi) = [c_t(\xi), c'_t(\xi)]$$

where $t = \lfloor \frac{k}{2} + 1 \rfloor$. The functions $c_{\lfloor \frac{k}{2}+1 \rfloor}$ and $c'_{\lfloor \frac{k}{2}+1 \rfloor}$ are known as the majority rule and dual majority rule, respectively. Thus L being finite

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and distributive implies that the median set for a given $\xi \in L^k$ is an order interval with bounds given by the majority and dual majority rule.

In 1990, Leclerc [8] proved that the converse holds. Specifically, for a finite lattice L, if the median set $M(\xi)$ is equal to $[c_{\lfloor \frac{k}{2}+1 \rfloor}(\xi), c'_{\lfloor \frac{k}{2}+1 \rfloor}(\xi)]$ for any $\xi \in L^k$, then L is distributive. Leclerc also proved that a finite lattice L is modular if and only if $M(\xi) \subseteq [c_{\lfloor \frac{k}{2}+1 \rfloor}(\xi), c'_{\lfloor \frac{k}{2}+1 \rfloor}(\xi)]$ for every $\xi \in L^k$. Moreover, he showed that L is upper semimodular if and only if $M(\xi) \subseteq [c_{\lfloor \frac{k}{2}+1 \rfloor}(\xi), 1_L]$ for every $\xi \in L^k$ where $1_L = \bigvee L$. The lower bound $c_{\lfloor \frac{k}{2}+1 \rfloor}(\xi)$ is tight as shown when L is distributive, but the upper bound of 1_L seems a bit crude and it is natural to ask for a better upper bound. Leclerc suggested the element

$$c_1(\xi) = \bigvee \{ \bigwedge_{i \in I} x_i : I \subseteq \{1, \ldots, k\}, |I| = 1\} = \bigvee_{i=1}^{k} x_i$$

as a possible upper bound for $M(\xi)$. In 2000, Li and Boukaabar [6] gave a nontrivial example of an upper semimodular lattice L with 101 elements in which there existed a $\xi \in L^3$ such that $c_1(\xi)$ was not an upper bound for $M(\xi)$. This example leads us to ask the following question. What conditions does a lattice L have to satisfy so that $c_1(\xi)$ does serve as an upper bound for $M(\xi)$ for any $\xi \in L^k$?

We say that the lattice L satisfies the c_1-*median property* if

$$\bigvee M(\xi) \leq c_1(\xi)$$

holds for all $\xi = (x_1, \ldots, x_k) \in L^k$. The motivation for the c_1-median property is the idea that this property may provide insight into the use of ordinal tools to help limit the search for medians. In this note we prove that a lattice of finite length satisfies the c_1-median property if it is graded and planar. Consequently, any planar upper semimodular lattice satisfies the c_1-median property. The class of slim semimodular lattices, which has been of interest in this journal [3], are known to be planar and so these lattices satisfy the c_1-median property as well.

2. Preliminaries

A lattice L is *graded* if any two maximal chains of L have the same number of elements. Let L be a graded lattice of finite length. For $x \in L$, the *height* $h(x)$ of x is equal to the length of the interval $[0_L, x]$ where $0_L = \bigwedge L$. Also, for $x, y \in L$, the classic distance between x and y in the undirected covering graph associated with L is denoted by $d(x, y)$. The graded condition imposes a structure that links $d(x, y)$,

$h(x)$, and $h(y)$. Namely, the following can be found as Lemma 2.1 in [5].

Lemma 2.1. *Let L be a graded lattice of finite length and let x and y be elements of L. Then*
(i) $d(x, y) \geq |h(x) - h(y)|$,
(ii) $d(x, y) = h(x) - h(y)$ *if and only if* $x \geq y$, *and*
(iii) $d(x, y) \geq |h(x) - h(y)| + 2$ *if* $x \parallel y$.

Leclerc made the following observation in the conclusion of his paper [8]. Suppose that L is a finite upper semimodular lattice, $\xi \in L^k$, and $m \in M(\xi)$. Leclerc asserted (without proof) that $h(m) \geq h(c_1(\xi))$ implies $m = c_1(\xi)$. The next Lemma gives a result that is similar to Leclerc's observation. However, we assume that L is a graded lattice of finite length.

Lemma 2.2. *Let L be a graded lattice of finite length. For any $\xi = (x_1, \ldots, x_k) \in L^k$ and for any $y \in L$ such that $y \neq c_1(\xi)$,*

$$h(y) \geq h(c_1(\xi)) \;\Rightarrow\; y \notin M(\xi).$$

Proof. Let L be a graded lattice of finite length, $\xi = (x_1, \ldots, x_k) \in L^k$, and let $x = c_1(\xi)$. Assume that $y \in L$ satisfies $h(y) \geq h(x)$ and $y \neq x$. Then, for each $x_i \in \xi$,

$$d(x, x_i) = h(x) - h(x_i) \leq h(y) - h(x_i) \leq d(y, x_i). \tag{2.1}$$

If $h(y) > h(x)$, then from (2.1) we get $d(x, x_i) < d(y, x_i)$ for all $x_i \in \xi$ and so $r(x, \xi) < r(y, \xi)$. Thus, $y \notin M(\xi)$. If $h(y) = h(x)$, then, since $y \neq x$, there exists $x_j \in \xi$ such that $x_j \not\leq y$. It follows from Lemma 2.1 that $d(y, x_j) > h(y) - h(x_j) = h(x) - h(x_j) = d(x, x_j)$. So then $d(x, x_j) < d(y, x_j)$ along with (2.1) imply that $r(x, \xi) < r(y, \xi)$. Again we have $y \notin M(\xi)$. □

We note that the converse of Lemma 2.2 does not hold. The lattice N_5 provides an example of a lattice that satisfies the conclusion of Lemma 2.2 that is not graded.

3. Main Result

A lattice L is *planar* if it has a planar Hasse diagram; see Kelly and Rival [7]. We now give the statement and proof of our main result.

Theorem 3.1. *Let L be a graded lattice of finite length. If L is planar, then L satisfies the c_1-median property.*

Proof. Let L be a graded lattice of finite length, $\xi = (x_1, \dots, x_k) \in L^k$, and let $x = c_1(\xi)$. We assume that a planar diagram of L is fixed. Suppose, for a contradiction, that $y \in L \setminus [0, x]$ but $y \in M(\xi)$. By Lemma 2.2, $h(y) < h(x)$. Hence, $y \parallel x$. Let C_0 and C_1 be the *left boundary chain* and the *right boundary chain* of $[0, x]$, respectively, in the fixed planar Hasse diagram of L; see Kelly and Rival [7]. They are maximal chains of $[0, x]$. Pick a maximal chain D in $[x, 1]$, and let $\overline{C}_i = C_i \cup D$. Since $y \parallel x$, we know from Propositions 1.6 and 1.7 of Kelly and Rival [7] that either y is strictly on the left of every maximal chain containing x, or y is strictly on the right of all these maximal chains. Hence, by left-right symmetry, we can assume that y is strictly on the left of $\overline{C}_0$.

For $i \in \{1, \dots, k\}$, take a path of length $d(y, x_i)$ from y to x_i in the covering graph of L. Further, the work found in [7] implies that this path contains an element $z_i \in \overline{C}_0$. We can assume that $z_i \in C_0$, because otherwise $x_i \le x < z_i$ and Lemma 2.1 allows us to modify the path so that it goes through both x and z_i. Since the path in question is of minimal length, $d(y, x_i) = d(y, z_i) + d(z_i, x_i)$, for $i \in \{1, \dots, k\}$. Forming the sum of these equalities and denoting $(z_1, \dots, z_k)$ and $d(z_1, x_1) + \dots + d(z_k, x_k)$ by ζ and $D(\zeta, \xi)$, respectively, we obtain $r(y, \xi) = r(y, \zeta) + D(\zeta, \xi)$. Let z_1 be one of the largest components of ζ. If $z_1 < y$, then Lemma 2.1 and the triangle inequality give $r(z_1, \xi) \le r(z_1, \zeta) + D(\zeta, \xi) < r(y, \zeta) + D(\zeta, \xi) = r(y, \xi)$, which contradicts $y \in M(\xi)$. So, we can assume $z_1 \not< y$. Furthermore, since $y \not\le x$, $z_1 \parallel y$. Let $z \in C_0$ be the unique element of C_0 with $h(z) = h(y)$, and note that $\{z, z_1, \dots, z_k\}$ is a chain. By Lemma 2.1, $d(z, z_i) = |h(z) - h(z_i)| = |h(y) - h(z_i)| \le d(y, z_i)$ for all $i \in \{1, \dots, k\}$ and $d(z, z_1) = |h(z) - h(z_1)| = |h(y) - h(z_1)| < d(y, z_1)$, since $z_1 \parallel y$. Combining these inequalities, $r(z, \zeta) < r(y, \zeta)$. Thus, $r(z, \xi) \le r(z, \zeta) + D(\zeta, \xi) < r(y, \zeta) + D(\zeta, \xi) = r(y, \xi)$, contradicting $y \in M(\xi)$. $\square$

The dual of Proposition 5.1 in [8] says that if a finite lattice L is lower semimodular, then for any $\xi \in L^k$ and for any $m \in M(\xi)$ the inequality $m \le c'_{\lfloor \frac{k}{2} + 1 \rfloor}(\xi)$ holds. Since $c'_{\lfloor \frac{k}{2} + 1 \rfloor}(\xi) \le c_1(\xi)$ for any $\xi \in L^k$, we can combine the dual of Proposition 5.1 in [8] with our main result to get the following corollary.

Corollary 3.2. *If L is a finite graded lattice that is planar or lower semimodular, then L satisfies the c_1-median property.*

Finally, note that Theorem 3.1 and its dual lead to the following result.

Corollary 3.3. *Suppose L is a finite lattice. If L is both graded and planar, then*

$$M(\xi) \subseteq [c_1'(\xi), c_1(\xi)]$$

for any $\xi = (x_1, \ldots, x_k) \in L^k$.

4. Concluding remarks

In this note, we have shown that a lattice L of finite length satisfies the c_1-median property if L is both planar and graded. These conditions are sufficient but not necessary. Indeed, if L is distributive and nonplanar or if L is the ungraded and planar lattice N_5, then L satisfies the c_1-median property. On the other hand, the following simple example shows why we can't stray too far from the graded condition. Let $L = \{0 = x_1, a_1, a_2, a_3, a_4 = x_2, y, 1\}$ be the 7-element lattice with $a_1 < \cdots < a_4$ and $y \parallel a_i$ for $i \in \{1, \ldots, 4\}$. If $\xi = (x_1, x_2)$, then it is easy to check that $M(\xi) = \{x_1, x_2, y, 1\}$. Since $y \not\leq x_1 \vee x_2 = x_2$ it follows that L does not satisfy the c_1-median property. The simplest example we know of a graded and nonplanar lattice L such that L does not satisfy the c_1-median property is the example given in [6]. Moreover, White [12] showed that if L is upper semimodular and L does not satisfy the c_1-median property, then the height of L is at least 7. Therefore, it would be interesting to uncover the precise connection between upper semimodularity and the c_1-median property.

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