Characterizing fully principal congruence representable distributive lattices

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Dedicated to the memory of Bjarni Jónsson

Abstract. Motivated by a recent paper of G. Grätzer, a finite distributive lattice D is called *fully principal congruence representable* if for every subset Q of D containing 0, 1, and the set $J(D)$ of nonzero join-irreducible elements of D, there exists a finite lattice L and an isomorphism from the congruence lattice of L onto D such that Q corresponds to the set of principal congruences of L under this isomorphism. A separate paper of the present author, see arXiv:1705.10833, contains a necessary condition of full principal congruence representability: D should be planar with at most one join-reducible coatom. Here we prove that this condition is sufficient. Furthermore, even the automorphism group of L can arbitrarily be stipulated in this case. Also, we generalize a recent result of G. Grätzer on principal congruence representable subsets of a distributive lattice whose top element is join-irreducible by proving that the automorphism group of the lattice we construct can be arbitrary.

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1. Introduction and our main goal

Unless otherwise specified explicitly, all lattices in this paper are assumed to be finite, even if this is not repeated all the time. For a finite lattice L, $J(L)$ denotes the ordered set of nonzero join-irreducible elements of L, $J_0(L)$ stands for $J(L) \cup \{0\}$, and we let $J^+(L) = J(L) \cup \{0, 1\}$. Also, $\mathrm{Princ}(L)$ denotes the *ordered set of all principal congruences* of L; it is a subset of the *congruence lattice* $\mathrm{Con}(L)$ of L and a superset of $J^+(\mathrm{Con}(L))$. It is well known that

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Con(L) is distributive. These facts motivate the following concept, which is due to Grätzer [15] and Grätzer and Lakser [19].

Definition 1.1. Let D be a finite distributive lattice. A subset $Q \subseteq D$ or, to be more precise, the inclusion $Q \subseteq D$ is *principal congruence representable* if there exist a finite lattice L and an isomorphism $\varphi\colon \mathrm{Con}(L) \to D$ such that $Q = \varphi(\mathrm{Princ}(L))$. We say that D is *fully principal congruence representable* if all subsets Q of D with $J^+(D) \subseteq Q$ are principal congruence representable.

Note that Czédli [8] uses the terminology "fl-representable" to indicate that L is *finite* and it is a *lattice*. We introduce a seemingly stronger property of D as follows. The *automorphism group* of a lattice L will be denoted by Aut(L).

Definition 1.2. A finite distributive lattice D is

fully principal congruence representable
with arbitrary automorphism groups, (1.1)

in short, (1.1)-*representable*, if for each subset Q of D such that $J^+(D) \subseteq Q$ and for any finite group G such that $|D| = 1 \Rightarrow |G| = 1$, there exist a finite lattice L and an isomorphism $\varphi\colon \mathrm{Con}(L) \to D$ such that $Q = \varphi(\mathrm{Princ}(L))$ and Aut(L) is isomorphic to G.

For more about full principal congruence representability, the reader can see Czédli [8], Grätzer [15] and Grätzer and Lakser [19]. The present paper relies on these papers, in particular, it depends heavily on Grätzer [15]. For related results on the representability of the ordered set Q as Princ(L) (without taking care of D), see Czédli [3], [4], [5], [6], and [7] and Grätzer [12], [14], [16], and [17].

Our main goal is to prove the following theorem.

Theorem 1.3 (Main Theorem). *If a finite distributive lattice is planar and contains at most one join-reducible coatom, then it is* (1.1)*-representable.*

The title of the present paper is motivated by the following statement, which will be concluded from Theorem 1.3, Czédli [8], and Grätzer [15] only in few lines.

Corollary 1.4. *If D is a finite distributive lattice, then the following three conditions are equivalent.*

(i) *D is fully principal congruence representable.*
(ii) *D is* (1.1)*-representable.*
(iii) *D is planar and it has at most one join-reducible coatom.*

Clearly, Corollary 1.4 and Czédli [8, Proposition 1.6] imply the following statement; the definition of full chain-representability is postponed to the next section.

Corollary 1.5. *A finite distributive lattice is fully principal congruence representable if and only if it is fully chain-representable.*

Outline

In Section 2, we recall the main result of Grätzer [15] as Theorem 2.1 in this paper, and we state its generalization in Theorem 2.2. Section 3 explains the construction required by the (iii) $\Rightarrow$ (i) part of Corollary 1.4 in a "proof-by-picture" way. Even if Section 3 contains no rigorous proof, it can rapidly convince the reader that our construction is "likely to work". In Section 4, we recall the quasi-coloring technique from Czédli [2] and develop it a bit further. In Section 5, armed with quasi-colorings, the "proof-by-picture" of Section 3 is turned to a rigorous proof of the implication 1.4(iii) $\Rightarrow$ 1.4(i). Section 6 contains a new proof of the hard part of G. Grätzer's Theorem 2.1. Section 7 modifies this proof to verify Theorem 2.2, and completes the proof of Theorem 1.3 and that of Corollary 1.4.

2. G. Grätzer's theorem and our second goal

For brevity, a subset Q of a finite distributive lattice D will be called a *candidate subset* if $J^+(D) \subseteq Q$. By a $J(D)$*-labeled chain* we mean a triplet $\langle C, \mathrm{lab}, D\rangle$ such that C is a finite chain, D is a finite distributive lattice, and

$$\mathrm{lab}\colon \mathrm{Prime}(C) \to J(D) \text{ is a surjective map from the } \textit{set } \mathrm{Prime}(C) \textit{ of all prime intervals} \text{ of } C \text{ onto } J(D). \tag{2.1}$$

Note that Grätzer [15] uses the terminology "$J(D)$-colored" rather than "$J(D)$-labeled" but here by a "coloring" we shall mean a particular *quasi-coloring*, which goes back to Czédli [2]. According to our terminology, the map in (2.1) is not a coloring in general. If $\mathfrak{p} \in \mathrm{Prime}(C)$, then $\mathrm{lab}(\mathfrak{p})$ is the *label* of the edge $\mathfrak{p}$. Given a $J(D)$-labeled chain $\langle C, \mathrm{lab}, D\rangle$, we define a map denoted by erep from the *set* $\mathrm{Intv}(C)$ *of all intervals* of C onto D as follows: for $I \in \mathrm{Intv}(C)$, let

$$\mathrm{erep}(I) := \bigvee_{\mathfrak{p}\in\mathrm{Prime}(I)} \mathrm{lab}(\mathfrak{p})\,; \tag{2.2}$$

the join is taken in D and $\mathrm{erep}(I)$ is called the *element represented* by I. The set

$$\mathrm{SRep}(C, \mathrm{lab}, D) := \{\mathrm{erep}(I) : I \in \mathrm{Intv}(C)\} \tag{2.3}$$

will be called the *set represented* by the $J(D)$-labeled chain $\langle C, \mathrm{lab}, D\rangle$. Clearly, $\mathrm{SRep}(C, \mathrm{lab}, D)$ is a candidate subset of D in this case. A candidate subset Q of D is said to be *chain-representable* if there exists a $J(D)$-labeled chain $\langle C, \mathrm{lab}, D\rangle$ such that $Q = \mathrm{SRep}(C, \mathrm{lab}, D)$. Note that C need not be a subchain of D.

If $1_D \in J(D)$ and $\langle C, \mathrm{lab}, D\rangle$ is $J(D)$*-labeled chain*, then we define a larger $J(D)$*-labeled chain* $\langle C^*, \mathrm{lab}^*, D\rangle$ as follows:

$$\text{we add a new largest element } 1_{C^*} \text{ to } C \text{ to obtain } C^* = C \cup \{1_{C^*}\} \text{ and we extend lab to lab}^* \text{ such that } \mathrm{lab}^*([1_C, 1_{C^*}]) = 1_D. \tag{2.4}$$

For elements x, y and a prime interval $\mathfrak{p}$ of a lattice L, $\mathrm{con}_L(x, y)$ and $\mathrm{con}_L(\mathfrak{p})$ denote the *congruence generated* by $\langle x, y\rangle$ and $\langle 0_{\mathfrak{p}}, 1_{\mathfrak{p}}\rangle$, respectively. The subscript is often dropped and we write $\mathrm{con}(x, y)$ and $\mathrm{con}(\mathfrak{p})$. A lattice

L will be called $\{0,1\}$-*separating* if for every $x \in L \setminus \{0,1\}$, $\text{con}(0,x) = \text{con}(x,1)$ is $1_{\text{Con}(L)}$, the largest congruence of L. The following result is due to Grätzer [15]; note that its part (iii) is implicit in [15], but the reader can find it by analyzing the construction given in [15]. For a different approach, see the proof of Theorem 2.2 here.

Theorem 2.1 (Grätzer [15]). *Let D be a finite distributive lattice. If Q is a candidate subset of D, that is, if $J^+ \subseteq Q \subseteq D$, then the following two statements hold.*

(i) *If $Q \subseteq D$ is principal congruence representable, then it is chain-representable.*
(ii) *If $1 = 1_D$ is join-irreducible and $Q \subseteq D$ is chain-representable, then $Q \subseteq D$ is principal congruence representable.*

Furthermore, if $Q \subseteq D$ is chain-representable and $1_D \in J(D)$, then

(iii) *for every $J(D)$-labeled chain $\langle C, \text{lab}, D\rangle$ representing $Q \subseteq D$, there exist a finite $\{0,1\}$-separating lattice L and an isomorphism $\varphi\colon \text{Con}(L) \to D$ such that*
  (a) $\varphi(\text{Princ}(L)) = \text{SRep}(C, \text{lab}, D) = Q$,
  (b) C^*, *defined in* (2.4), *is a filter of L,*
  (c) $\text{lab}^*(\mathfrak{p}) = \varphi(\text{con}_L(\mathfrak{p}))$ *holds for every* $\mathfrak{p} \in \text{Prime}(C^*)$, *and*
  (d) *for all $x \in C^*$ and $y \in L \setminus C^*$, if $y \prec x$, then* $\text{con}_L(y,x) = 1_{\text{Con}(L)}$.

For the $1_D \in J(D)$ case, we are going to generalize Theorem 2.1(iii) as follows; note that if we did not care with $\text{Aut}(L)$, then our lattice L would often be smaller than the corresponding lattice constructed in Grätzer [15].

Theorem 2.2. *Let D be a finite distributive lattice such that $1 = 1_D$ is join-irreducible and $|D| > 1$, let G be a finite group, and let Q be candidate subset of D. If $Q \subseteq D$ is chain-representable, then for every $J(D)$-labeled chain $\langle C, \text{lab}, D\rangle$ that represents Q, there exist a finite $\{0,1\}$-separating lattice L and an isomorphism $\varphi\colon \text{Con}(L) \to D$ such that*

(i) $\text{SRep}(C, \text{lab}, D) = \varphi(\text{Princ}(L)) = Q$,
(ii) C^*, *which is defined in* (2.4), *is a filter of L,*
(iii) $\text{lab}^*(\mathfrak{p}) = \varphi(\text{con}_L(\mathfrak{p}))$ *holds for every* $\mathfrak{p} \in \text{Prime}(C^*)$,
(iv) *for all $x \in C^*$ and $y \in L \setminus C^*$, if $y \prec x$, then* $\text{con}_L(y,x) = 1_{\text{Con}(L)}$, *and*
(v) $\text{Aut}(L)$ *is isomorphic to G.*

Remark 2.3. The proof will make it clear that Theorem 2.2 remains true if we replace "finite group" and "finite $\{0,1\}$-separating lattice L" by "group" and "$\{0,1\}$-separating lattice L of finite length", respectively. No further details of this fact will be given later.

3. From $1 \in J(D)$ to $1 \notin J(D)$, a proof-by-picture approach

In this section, we outline our construction that derives the (iii) $\Rightarrow$ (i) part of Corollary 1.4 from Theorem 2.1. Since the $1 \in J(D)$ case follows from the

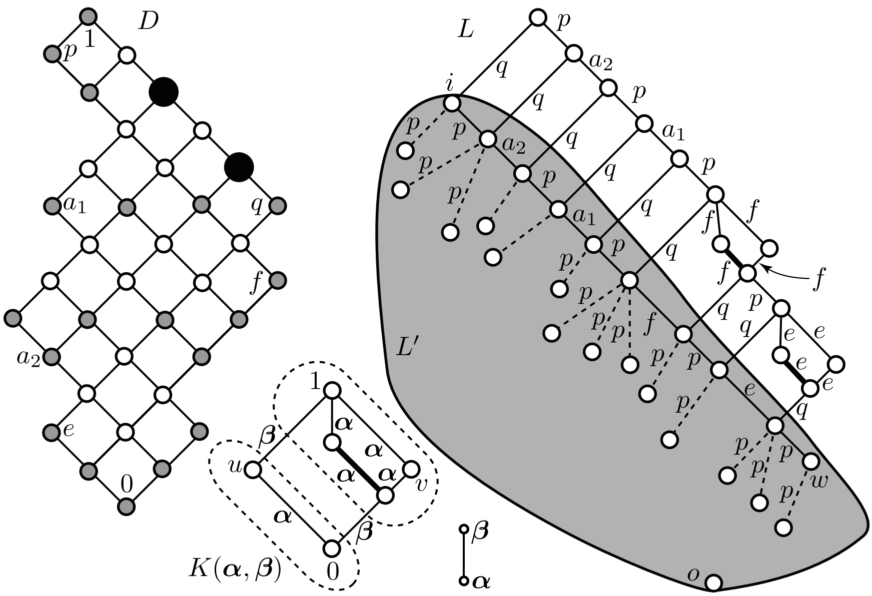

FIGURE 1. D with two coatoms, $K(\boldsymbol{\alpha},\boldsymbol{\beta})$, and L that we construct

conjunction of Czédli [8, Proposition 1.6] and Grätzer [15], here we deal only with the case where $1 = 1_D$ is join-reducible. So, in this section, we assume that D is a planar distributive lattice such that $1 = 1_D$ is join-reducible. It belongs to the folklore that

every element x of D covers at most two elements and x is the join of at most two join-irreducible elements; (3.1)

see, for example, Czédli [8, (2.1) and (2.3)] or Grätzer and Knapp [18]. We assume conditions (iii) of Corollary 1.4; in particular, D has at most one join-reducible coatom. Hence (3.1) yields that there are distinct $p, q \in J(D)$ such that $1_D = p \vee q$ and $p \prec 1$; see Figure 1. (In Section 5, there will be more explanation of this fact and other facts we are going to assert.) Also, let $Q \subseteq D$ such that $J^+(D) \subseteq Q$. In the figure, Q consists of the grey-filled and the large black-filled elements. Let us denote by D' the principal ideal ${\downarrow}p = \{d \in D : d \leq p\}$, and let $Q' = Q \cap D'$. It will not be hard to show that

the filter ${\uparrow}q = \{d \in D : d \geq q\}$ is a chain, D is the disjoint union of D' and ${\uparrow}q$, and q is a maximal element of $J(D)$, (3.2)

as shown in the figure. Next, we focus on $(Q \cap {\uparrow}q) \setminus \{q\}$; it consists of the large black-filled elements in the figure. By the maximality of q in $J(D)$, these elements are join-reducible, whereby each of them is the join of q and another join-irreducible element a_i. In our case, $(Q \cap {\uparrow}q) \setminus \{q\} = \{a_1 \vee q, a_2 \vee q\}$; in general, it is $\{a_1, \ldots, a_k\}$ where $k \geq 0$. We will show that

$$J(D') \cap {\downarrow}q \text{ has at most two maximal elements.} \tag{3.3}$$

Let $\{e,f\}$ be the set of maximal elements of $J(D')\cap{\downarrow}q$; note that $e=f$ is possible but causes no problem.

Since $p=1_{D'}$ is join-irreducible, D' has only one coatom. Hence, we know from Czédli [8, Proposition 1.6] that $Q'\subseteq D'$ is represented by a $J(D')$-labeled chain $\langle C_0, \mathrm{lab}'_0, D'\rangle$. Let C_1 be the chain of length $2k+4=8$ whose edges, starting from below, are colored by p,e,p,f,p,a_1,p,a_2. The

glued sum $C := C_0 \dot{+} C_1$ is obtained from their summands by putting C_1 atop C_0 and identifying the top element of C_0 with the bottom element of C_1. (3.4)

In this way, we have obtained a $J(D')$-labeled chain $\langle C, \mathrm{lab}', D'\rangle$. It will be easy to show that

$$\langle C, \mathrm{lab}', D'\rangle \text{ also represents } Q'\subseteq D'. \tag{3.5}$$

Therefore, Theorem 2.1 yields a finite lattice L' and a lattice isomorphism $\varphi'\colon \mathrm{Con}(L')\to D'$ such that 2.1(iii) holds with $\langle Q', D', C, L', \varphi'\rangle$ instead of $\langle Q, D, C, L, \varphi\rangle$. In particular,

$$\varphi'(\mathrm{Princ}(L')) = Q'. \tag{3.6}$$

In the figure, L' is represented by the grey-filled area on the right. In C, there is a unique element w such that $C_0={\downarrow}w$ and $C_1={\uparrow}w$ (understood in C, not in L'). Only ${\uparrow}w$, which is a filter of L' and also a filter of C^*, see (2.4), is indicated in the figure. Since $p=1_{D'}$, the top edge of ${\uparrow_{L'}}\,w$ (the filter understood in L') is p-labeled. Some elements outside C^* that are covered by elements of ${\uparrow_{L'}}\,w$ are also indicated in the figure; the covering relation in these cases are shown by dashed lines; 2.1(iiid) and $p=1_{D'}$ motivate that these edges are labeled by p.

Next, by adding $2k+9=13$ new elements to L', we obtain a larger lattice L, as indicated in Figure 1. For a congruence $\boldsymbol{\gamma}\in\mathrm{Con}(L')$, let $\mathrm{con}_L(\boldsymbol{\gamma})$ denote the congruence of L that is generated by the relation $\boldsymbol{\gamma}\subseteq L^2$. Each of the edges labeled by q generate the same congruence, which we denote by $\widehat{q}\in\mathrm{Con}(L)$. Consider the lattice $K(\boldsymbol{\alpha},\boldsymbol{\beta})$ in the middle of Figure 1. For later reference, note that the only property of this lattice that we will use is that

$$K(\boldsymbol{\alpha},\boldsymbol{\beta}) \text{ has exactly one nontrivial congruence,} \tag{3.7}$$

$\boldsymbol{\alpha}$, whose blocks are indicated by dashed ovals. Hence, if this lattice is a sublattice of L, then any of its $\boldsymbol{\alpha}$-colored edge generates a congruence that is smaller than or equal to the congruence generated by a $\boldsymbol{\beta}$-colored edge. Copies of this lattice ensure the following two "comparabilities"

$$\mathrm{con}_L(\varphi'^{-1}(e))\le\widehat{q} \text{ and } \mathrm{con}_L(\varphi'^{-1}(f))\le\widehat{q}. \tag{3.8}$$

For $x\parallel y\in L$, if x and y cover their meet and are covered by their join, then $\{x\wedge y, x, y, x\vee y\}$ is a *covering square* of L. For $i\in 1,\dots,k=\{1,2\}$,

the covering square with a_i,q,a_i,q-labeled edges guarantees that $\mathrm{con}_L(\varphi'^{-1}(a_i))\vee\widehat{q}$ is a principal congruence of L. (3.9)

The map $\varphi\colon \mathrm{Con}(L) \to D$ we are going to define will satisfy the rule

$$\varphi(\mathrm{con}_L(\varphi'^{-1}(x))) = x \text{ for } x \in D' \text{ and } \varphi(\widehat{q}) = q. \tag{3.10}$$

It will be easy to see that (3.10) determines φ uniquely and that our construction yields all comparabilities and principal congruences that we need. We will rigorously prove that we do not get more comparabilities and principal congruences than those described in (3.8) and (3.9). Thus, it will be straightforward to conclude the (iii) $\Rightarrow$ (i) part of Corollary 1.4

4. Quasi-colored lattices

Reflexive and transitive relations are called *quasiorderings*, also known as *preorderings*. If ν is a quasiordering on a set A, then $\langle A;\nu\rangle$ is said to be a *quasiordered set*. For $H \subseteq A^2$, the least quasiordering of A that includes H will be denoted by $\mathrm{quo}_A(H)$, or simply by $\mathrm{quo}(H)$ if there is no danger of confusion. For $H = \{\langle a,b\rangle\}$, we will of course write $\mathrm{quo}(a,b)$. Quite often, especially if we intend to exploit the transitivity of ν, we write $a \leq_\nu b$ or $b \geq_\nu a$ instead of $\langle a,b\rangle \in \nu$. Also, $a =_\nu b$ will stand for $\{\langle a,b\rangle, \langle b,a\rangle\} \subseteq \nu$. The set of all quasiorderings on A form a complete lattice $\mathrm{Quo}(A)$ under set inclusion. For $\nu,\tau \in \mathrm{Quo}(A)$, the join $\nu \vee \tau$ is $\mathrm{quo}(\nu \cup \tau)$. *Orderings* are antisymmetric quasiorderings, and a set with an ordering is an *ordered set*, also known as a *poset*. Following Czédli [2], a *quasi-colored lattice* is a lattice L of finite length together with a surjective map γ, called a *quasi-coloring*, from $\mathrm{Prime}(L)$ onto a quasiordered set $\langle H;\nu\rangle)$ such that for all $\mathfrak{p},\mathfrak{q} \in \mathrm{Prime}(L)$,

(C1) if $\gamma(\mathfrak{p}) \geq_\nu \gamma(\mathfrak{q})$, then $\mathrm{con}(\mathfrak{p}) \geq \mathrm{con}(\mathfrak{q})$, and
(C2) if $\mathrm{con}(\mathfrak{p}) \geq \mathrm{con}(\mathfrak{q})$, then $\gamma(\mathfrak{p}) \geq_\nu \gamma(\mathfrak{q})$.

The values of γ are called *colors* (rather than quasi-colors). If $\gamma(\mathfrak{p}) = b$, then we say that $\mathfrak{p}$ is colored by b. In figures, the colors of (some) edges are indicated by labels. Note the difference: even if the colors are often given by labels, a labeling like (2.1) need not be a quasi-coloring. If $\langle H;\nu\rangle$ happens to be an ordered set, then γ above is a *coloring*, not just a quasi-coloring. The map γ_{nat} from $\mathrm{Prime}(L)$ to $J(\mathrm{Con}(L)) = \langle J(\mathrm{Con}(L)); \leq\rangle$, defined by $\gamma_{\mathrm{nat}}(\mathfrak{p}) := \mathrm{con}(\mathfrak{p})$, is the so-called *natural coloring* of L. The relevance of quasi-colorings of a lattice L of finite length lies in the fact that they determine $\mathrm{Con}(L)$; see Czédli [2, (2.8)]. Even if we will use quasi-colorings in our stepwise constructing method, we need only the following statement. For convenience, we present its short proof here rather than explaining how to extract the statement from Czédli [2].

Lemma 4.1. *Let L and D be a finite lattice and a finite distributive lattice, respectively. If $\widehat{\gamma}\colon \mathrm{Prime}(L) \to J(D)$ is a coloring, then the map*

$$\mu\colon \langle J(\mathrm{Con}(L)); \leq\rangle \to \langle J(D); \leq\rangle,\ \textit{defined by } \mathrm{con}(\mathfrak{p}) \mapsto \widehat{\gamma}(\mathfrak{p})$$

where $\mathfrak{p} \in \mathrm{Prime}(L)$, is an order isomorphism.

Proof. It is well known that

$$J(\mathrm{Con}(L)) = \{\mathrm{con}(\mathfrak{p}) : \mathfrak{p} \in \mathrm{Prime}(L)\}. \tag{4.1}$$

We obtain from (C2) that μ is well defined, that is, if $\mathrm{con}(\mathfrak{p}) = \mathrm{con}(\mathfrak{q})$, then $\widehat{\gamma}(\mathfrak{p}) = \widehat{\gamma}(\mathfrak{q})$. Furthermore, (C2) gives that μ is order-preserving. It is surjective since so is $\widehat{\gamma}$. We conclude from (C1) that $\mu(\mathrm{con}(\mathfrak{p})) \leq \mu(\mathrm{con}(\mathfrak{q}))$ implies that $\mathrm{con}(\mathfrak{p}) \leq \mathrm{con}(\mathfrak{q})$. This also yields that μ is injective. □

Next, assume that $\langle A_1; \nu_1\rangle$ and $\langle A_2; \nu_2\rangle$ are quasiordered sets. By a *homomorphism* $\delta\colon \langle A_1; \nu_1\rangle \to \langle A_2; \nu_2\rangle$ we mean a map $\delta\colon A_1 \to A_2$ such that $\delta(\nu_1) \subseteq \nu_2$, that is, $\langle \delta(x), \delta(y)\rangle \in \nu_2$ holds for all $\langle x, y\rangle \in \nu_1$. Following Czédli and Lenkehegyi [10],

$$\vec{\mathrm{Ker}}(\delta) := \bigl\{\langle x, y\rangle \in A_1^2 : \bigl(g(x), g(y)\bigr) \in \nu_2\bigr\} \tag{4.2}$$

is called the *directed kernel* of δ. Clearly, it is a quasiordering on A_1 for an arbitrary map $\delta\colon A_1 \to A_2$, which need not be a homomorphism. Note that δ is a homomorphism if and only if $\vec{\mathrm{Ker}}(\delta) \supseteq \nu_1$. The following lemma, which we need later, is Lemma 2.1 in Czédli [2]. Note that we compose maps from right to left.

Lemma 4.2 ([2]). *Let M be a finite lattice, and let $\langle Q; \nu\rangle$ and $\langle P; \sigma\rangle$ be quasiordered sets. Let $\gamma_0\colon \mathrm{Prime}(M) \to \langle Q; \nu\rangle$ be a quasi-coloring. Let us assume that $\delta\colon \langle Q; \nu\rangle \to \langle P; \sigma\rangle$ is a surjective homomorphism such that $\vec{\mathrm{Ker}}(\delta) \subseteq \nu$. Then the composite map $\delta \circ \gamma_0\colon \mathrm{Prime}(M) \to \langle P; \sigma\rangle$ is a quasi-coloring.*

The advantage of quasi-colorings over colorings is that, as opposed to orderings, quasiorderings form a lattice; see Czédli [2, p. 315] for more motivation. We are going to prove and use the following lemma, which gives even more motivation. If L_1 is an ideal and L_2 is a filter of a lattice L such that $L_1 \cup L_2 = L$ and $L_1 \cap L_2 \neq \emptyset$, then L is the (*Hall–Dilworth*) *gluing* of L_1 and L_2 over their intersection.

Lemma 4.3. *Let L be a lattice of finite length such that it is the Hall–Dilworth gluing of L_1 and L_2 over $L_1 \cap L_2$. For $i \in \{1, 2\}$, let $\gamma_i\colon \mathrm{Prime}(L_i) \to \langle H_i; \nu_i\rangle$ be a quasi-coloring, and assume that*

$$H_1 \cap H_2 \subseteq \{\gamma_1(\mathfrak{p}) : \mathfrak{p} \in \mathrm{Prime}(L_1 \cap L_2) \text{ and } \gamma_1(\mathfrak{p}) = \gamma_2(\mathfrak{p})\}. \tag{4.3}$$

Let $H := H_1 \cup H_2$, and define $\gamma\colon \mathrm{Prime}(L) \to H$ by the rule

$$\gamma(\mathfrak{p}) = \begin{cases} \gamma_1(\mathfrak{p}) & \text{for } \mathfrak{p} \in \mathrm{Prime}(L_1), \\ \gamma_2(\mathfrak{p}) & \text{for } \mathfrak{p} \in \mathrm{Prime}(L) \setminus \mathrm{Prime}(L_1). \end{cases} \tag{4.4}$$

Let

$$\nu = \mathrm{quo}\bigl(\nu_1 \cup \nu_2 \cup \{\langle \gamma_j(\mathfrak{p}), \gamma_{3-j}(\mathfrak{p})\rangle : \mathfrak{p} \in \mathrm{Prime}(L_1 \cap L_2),\ \ j \in \{1, 2\}\}\bigr). \tag{4.5}$$

Then $\gamma\colon \mathrm{Prime}(L) \to \langle H; \nu\rangle$ is a quasi-coloring.

Note the following three facts. In (4.4), the subscripts 1 and 2 could be interchanged and even a "mixed" definition of $\gamma(\mathfrak{p})$ would work. Even if γ_1 and γ_2 are colorings, γ in Lemma 4.3 is only a quasi-coloring in general. The case where $|L_1| = |L_2| = 2 = |L| - 1$ and $H_1 = H_2$ exemplifies that the assumption (4.3) cannot be omitted.

Before proving the lemma, we recall a useful statement from Grätzer [13]. For $i \in \{1, 2\}$, let $\mathfrak{p}_i = [x_i, y_i]$ be a prime interval of a lattice L. We say that $\mathfrak{p}_1$ is *prime-perspective down* to $\mathfrak{p}_2$, denoted by $\mathfrak{p}_1 \overset{\text{p-dn}}{\to} \mathfrak{p}_2$ or $\langle x_1, y_1\rangle \overset{\text{p-dn}}{\to} \langle x_2, y_2\rangle$, if $y_1 = x_1 \vee y_2$ and $x_1 \wedge y_2 \le x_2$; see Figure 2, where the solid lines indicate prime intervals while the dotted ones stand for the ordering relation of L. We define *prime-perspective up*, denoted by $\mathfrak{p}_1 \overset{\text{p-up}}{\to} \mathfrak{p}_2$, dually. The reflexive transitive closure of the union of $\overset{\text{p-up}}{\to}$ and $\overset{\text{p-dn}}{\to}$ is called *prime-projectivity.*

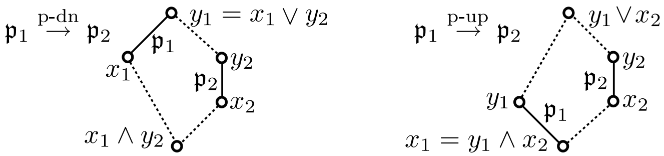

FIGURE 2. Prime perspectivities

Lemma 4.4 (Prime-Projectivity Lemma; see Grätzer [13]**).** *Let L be a lattice of finite length, and let $\mathfrak{r}_1$ and $\mathfrak{r}_2$ be prime intervals in L. Then we have that* $\operatorname{con}(\mathfrak{r}_1) \ge \operatorname{con}(\mathfrak{r}_2)$ *if and only if there exist an $n \in \mathbb{N}_0$ and a sequence $\mathfrak{r}_1 = \mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_n = \mathfrak{r}_2$ of* prime *intervals such that for each $i \in \{1, \dots, n\}$, $\mathfrak{p}_{i-1} \overset{\text{p-dn}}{\to} \mathfrak{p}_i$ or $\mathfrak{p}_{i-1} \overset{\text{p-up}}{\to} \mathfrak{p}_i$.*

Proof of Lemma 4.3. In order to prove (C1), let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals of L such that $\gamma(\mathfrak{p}) \ge_\nu \gamma(\mathfrak{q})$. By the definition of ν, there is a sequence $\gamma(\mathfrak{p}) = h_0, h_1, h_2, \dots, h_k = \gamma(\mathfrak{q})$ in H such that, for each i, $h_{i-1} \ge_{\nu_1} h_i$, or $h_{i-1} \ge_{\nu_2} h_i$, or $\langle h_{i-1}, h_i\rangle = \langle \gamma_j(\mathfrak{r}_{i-1}), \gamma_{3-j}(\mathfrak{r}'_i)\rangle$ for some $j = j(i) \in \{1, 2\}$ and $\mathfrak{r}_{i-1} = \mathfrak{r}'_i \in \operatorname{Prime}(L_1 \cap L_2) = \operatorname{Prime}(L_1) \cap \operatorname{Prime}(L_2)$. In the first case, by the surjectivity of γ_1 and the satisfaction of (C1) in L_1, we can pick prime intervals $\mathfrak{r}_{i-1}, \mathfrak{r}'_i \in \operatorname{Prime}(L_1)$ such that $\gamma_1(\mathfrak{r}_{i-1}) = h_{i-1}$, $\gamma_1(\mathfrak{r}'_i) = h_i$, and $\operatorname{con}_{L_1}(\mathfrak{r}_{i-1}) \ge \operatorname{con}_{L_1}(\mathfrak{r}'_i)$. In the second case, we obtain similarly that $\gamma_2(\mathfrak{r}_{i-1}) = h_{i-1}$, $\gamma_2(\mathfrak{r}'_i) = h_i$, and $\operatorname{con}_{L_2}(\mathfrak{r}_{i-1}) \ge \operatorname{con}_{L_2}(\mathfrak{r}'_i)$ for some $\mathfrak{r}_{i-1}, \mathfrak{r}'_i \in \operatorname{Prime}(L_2)$. In the third case, both $\operatorname{con}_{L_1}(\mathfrak{r}_{i-1}) \ge \operatorname{con}_{L_1}(\mathfrak{r}'_i)$ and $\operatorname{con}_{L_2}(\mathfrak{r}_{i-1}) \ge \operatorname{con}_{L_2}(\mathfrak{r}'_i)$ trivially hold, since $\mathfrak{r}_{i-1} = \mathfrak{r}'_i$.

Hence, for for every i in $\{1, \dots, k\}$, Lemma 4.4 gives us

$$\text{a "prime-projectivity sequence" from } \mathfrak{r}_{i-1} \text{ to } \mathfrak{r}'_i. \tag{4.6}$$

Since $\operatorname{Prime}(L_1) \subseteq \operatorname{Prime}(L)$ and $\operatorname{Prime}(L_2) \subseteq \operatorname{Prime}(L)$, this sequence is in $\operatorname{Prime}(L)$. We claim that, for each $i \in \{1, \dots, k\}$,

$$\text{there is a prime-projectivity sequence from } \mathfrak{r}'_i \text{ to } \mathfrak{r}_i. \tag{4.7}$$

In order to verify this, note that h_i is the color of $\mathfrak{r}_i$ with respect to γ_1 or γ_2, and it is also the color of $\mathfrak{r}'_i$ with respect to γ_1 or γ_2. Assume first that $\gamma_1(\mathfrak{r}'_i) = h_i = \gamma_1(\mathfrak{r}_i)$. Then $\gamma_1(\mathfrak{r}'_i) \geq_{\nu_1} \gamma_1(\mathfrak{r}_i)$ and the validity of (C1) for γ_1 imply that $\mathrm{con}_{L_1}(\mathfrak{r}'_i) \geq \mathrm{con}_{L_1}(\mathfrak{r}_i)$, whereby (4.7) follows from Lemma 4.4. The case $\gamma_2(\mathfrak{r}'_i) = h_i = \gamma_2(\mathfrak{r}_i)$ is similar. Hence, we can assume that $\gamma_j(\mathfrak{r}'_i) = h_i = \gamma_{3-j}(\mathfrak{r}_i)$ for some $j \in \{1, 2\}$. Clearly, h_i is in $H_1 \cap H_2$, since it is in the range of γ_j and that of γ_{3-j}. By (4.3), we can pick a prime interval $\mathfrak{r}''_i \in \mathrm{Prime}(L_1 \cap L_2)$ such that $\gamma_j(\mathfrak{r}''_i) = h_i = \gamma_{3-j}(\mathfrak{r}''_i)$. Applying (C1) to γ_j and Lemma 4.4, we obtain that there is a prime-projectivity sequence from $\mathfrak{r}'_i$ to $\mathfrak{r}''_i$. Similarly, we obtain a prime-projectivity sequence from $\mathfrak{r}''_i$ to $\mathfrak{r}_i$. Concatenating these two sequences, we obtain a prime-projectivity sequence from $\mathfrak{r}'_i$ to $\mathfrak{r}_i$. This shows the validity of (4.7). Finally, concatenating the sequences from (4.6) and those from (4.7), we obtain a prime-projectivity sequence from $\mathfrak{p} = \mathfrak{r}_0$ to $\mathfrak{q} = \mathfrak{r}_k$. So the easy direction of Lemma 4.4 implies that $\mathrm{con}(\mathfrak{p}) \geq \mathrm{con}(\mathfrak{q})$, proving that L satisfies (C1).

Observe that

$$\text{for all } i \in \{1, 2\} \text{ and } \mathfrak{r} \in \mathrm{Prime}(L_i),\ \gamma_i(\mathfrak{r}) =_\nu \gamma(\mathfrak{r}); \tag{4.8}$$

this is clear either because $\mathfrak{r} \notin \mathrm{Prime}(L_{3-i})$ and (4.4) applies, or because $\mathfrak{r}$ belongs to $\mathrm{Prime}(L_1 \cap L_2)$, $\gamma(\mathfrak{r}) = \gamma_1(\mathfrak{r})$ by (4.4), and we have by (4.5) that $\{\langle \gamma_1(\mathfrak{r}), \gamma_2(\mathfrak{r})\rangle, \langle \gamma_2(\mathfrak{r}), \gamma_1(\mathfrak{r})\rangle\} \subseteq \nu$.

Next, in order to prove that L satisfies (C2), assume that $\mathfrak{p}, \mathfrak{q} \in \mathrm{Prime}(L)$ such that $\mathrm{con}(\mathfrak{p}) \geq \mathrm{con}(\mathfrak{q})$. We need to show that $\gamma(\mathfrak{p}) \geq_\nu \gamma(\mathfrak{q})$. This is clear if $\mathfrak{p} = \mathfrak{q}$. Since ν is transitive, Lemma 4.4 and duality allow us to assume that $\mathfrak{p} \overset{\text{p-up}}{\to} \mathfrak{q}$. We are going to deal only with the case $\mathfrak{p} \in \mathrm{Prime}(L_1) \setminus \mathrm{Prime}(L_2)$ and $\mathfrak{q} \in \mathrm{Prime}(L_2) \setminus \mathrm{Prime}(L_1)$, since the cases $\{\mathfrak{p}, \mathfrak{q}\} \subseteq \mathrm{Prime}(L_1)$ and $\{\mathfrak{p}, \mathfrak{q}\} \subseteq \mathrm{Prime}(L_2)$ are much easier while the case $\mathfrak{p} \in \mathrm{Prime}(L_2) \setminus \mathrm{Prime}(L_1)$ and $\mathfrak{q} \in \mathrm{Prime}(L_1) \setminus \mathrm{Prime}(L_2)$ is excluded by the upward orientation of the prime-perspectivity. So $\mathfrak{p} = [x_1, y_1] = [y_1 \wedge x_2, y_1]$ and $\mathfrak{q} = [x_2, y_2]$ with $y_2 \leq y_1 \vee x_2$; see Figure 2. Clearly, $x_1, y_1 \in L_1 \setminus L_2$ and $x_2, y_2 \in L_2 \setminus L_1$. By the description of the ordering relation in Hall–Dilworth gluings, we can pick an $x_3 \in L_1 \cap L_2$ such that $x_1 \leq x_3 \leq x_2$. Let $y_3 := y_1 \vee x_3$. It is in $L_1 \cap L_2$ since L_1 is a sublattice and L_2 is a filter in L. Since $x_3 \vee x_2 = x_2 \leq y_2 \leq y_1 \vee x_2 = y_1 \vee (x_3 \vee x_2) = (y_1 \vee x_3) \vee x_2 = y_3 \vee x_2$, we have that $\mathrm{con}(\mathfrak{q}) = \mathrm{con}(x_2, y_2) \leq \mathrm{con}(x_3, y_3)$. Combining this inequality, (4.1), the distributivity of $\mathrm{Con}(L)$, the well-known rule that

$$\begin{aligned}&\text{in every finite distributive lattice } D,\\ &(a \in J(D) \text{ and } a \leq b_1 \vee \ldots \vee b_n) \Longrightarrow (\exists i)(a \leq b_i),\end{aligned} \tag{4.9}$$

and $\mathrm{con}(x_3, y_3) = \bigvee\{\mathrm{con}(\mathfrak{r}) : \mathfrak{r} \in \mathrm{Prime}([x_3, y_3])\}$, we obtain a prime interval $\mathfrak{r} \in \mathrm{Prime}([x_3, y_3]) \subseteq \mathrm{Prime}(L_1 \cap L_2)$ such that $\mathrm{con}(\mathfrak{q}) \leq \mathrm{con}(\mathfrak{r})$. Since $\mathrm{con}(\mathfrak{p}) = \mathrm{con}(x_1, y_1)$ collapses $\langle x_3, y_3\rangle = \langle x_1 \vee x_3, y_1 \vee x_3\rangle$, it collapses $\mathfrak{r}$. Hence, $\mathrm{con}(\mathfrak{p}) \geq \mathrm{con}(\mathfrak{r})$. Since γ_1 is a quasi-coloring, this inequality, (C2), and (4.8) yield that $\gamma(\mathfrak{p}) = \gamma_1(\mathfrak{p}) \geq_{\nu_1} \gamma_1(\mathfrak{r}) =_\nu \gamma(\mathfrak{r})$. Hence, $\gamma(\mathfrak{p}) \geq_\nu \gamma(\mathfrak{r})$. Since γ_2 is also a quasi-coloring, the already established $\mathrm{con}(\mathfrak{r}) \geq \mathrm{con}(\mathfrak{q})$ leads to $\gamma(\mathfrak{r}) = \gamma_2(\mathfrak{r}) \geq_{\nu_2} \gamma_2(\mathfrak{q}) =_\nu \gamma(\mathfrak{q})$ similarly, whereby $\gamma(\mathfrak{r}) \geq_\nu \gamma(\mathfrak{q})$.

Thus, transitivity gives that $\gamma(\mathfrak{p}) \geq_\nu \gamma(\mathfrak{q})$, showing that L satisfies (C2). This completes the proof of Lemma 4.3. □

5. Proving the (iii) ⇒ (i) part of Corollary 1.4

Although Corollary 1.4 will be a consequence of Theorem 1.3, here we are going to derive the (iii) ⇒ (i) part of this corollary from Theorem 2.1. In this way, this section serves as a part of the proof of Theorem 1.3.

Proof of the (iii) ⇒ (i) *part of Corollary* 1.4. Let D be an arbitrary planar distributive lattice with at most one join-reducible atom. We can assume that $|D| > 1$.

First, assume that $1_D \in J(D)$. We know from Czédli [8, Proposition 1.6] that for every Q, if $J^+(D) \subseteq Q \subseteq D$, then the inclusion $Q \subseteq D$ is chain-representable. Hence, by Theorem 2.1(ii), it fully is principal congruence representable, as required.

Second, assume that $1_D \notin J(D)$. Let Q be a subset of the lattice D such that $J^+(D) \subseteq Q$. In order to obtain a lattice L that witnesses the principal congruence representability of $Q \subseteq D$, we do the same as in Section 3 but, of course, now we cannot assume that $k = 2$ and we are going to give more details. By $1_D \notin J(D)$ and (3.1), there are exactly two coatoms. At least one of them is join-irreducible; we denote it by p. Since 1_D is a join of join-irreducible element, $J(D) \not\subseteq {\downarrow}p$. Thus, we can pick a maximal element q in (the nonempty set) $J(D) \setminus {\downarrow}p$ such that $1_D = p \vee q$. Clearly, both p and q are maximal elements of $J(D)$. Since D is a planar distributive lattice, we know from the folklore or, say, from Czédli and Grätzer [9] that

$$J(D) \text{ is the union of two chains.} \tag{5.1}$$

So we have two chains C_1 and C_2 such that $J_0(D) = C_1 \cup C_2$ and $0 \in C_1 \cap C_2$. Let, say, $q \in C_2$. If $x \in {\uparrow}q$ and $x \neq q$, then $x = y_1 \vee y_2$ for some $y_1 \in C_1$ and $y_2 \in C_2$. Since q is a maximal element of $J(D)$ and so also of C_2, $y_2 \leq q$ and $x = y_1 \vee q$. For $x = q$, we can let $y_1 = 0 \in C_1$. Hence, ${\uparrow}q \subseteq \{z \vee q : z \in C_1\}$. Since C_1 is a chain, so are $\{z \vee q : z \in C_1\}$ and its subset ${\uparrow}q$. Finally, $D' := {\downarrow}p$ is disjoint from ${\uparrow}q$ since $p \parallel q$. The facts established so far prove (3.2).

Observe that (5.1) implies (3.3). Note that even if $e = f$, we will construct L as given on the right of Figure 1. This will cause no problem since then f can be treated as an alter ego of e, similarly to the alter egos $p_1, \dots, p_{k+3}$, see later, of p.

In order to verify (3.5), observe that $\mathrm{Intv}(C_0) \subseteq \mathrm{Intv}(C)$ implies that the inclusion $\mathrm{SRep}(C_0, \mathrm{lab}'_0, D') \subseteq \mathrm{SRep}(C, \mathrm{lab}', D')$ holds; see (2.3) for the notation. To see the converse inclusion, let $I \in \mathrm{Intv}(C)$. If $\mathrm{length}(I) \leq 1$, then $\mathrm{erep}(I) \in J_0(D') \subseteq \mathrm{SRep}(C_0, \mathrm{lab}'_0, D')$ is clear. If $\mathrm{length}(I) \geq 2$, then either $I \in \mathrm{Intv}(C_0)$ and $\mathrm{erep}(I) \in \mathrm{SRep}(C_0, \mathrm{lab}'_0, D')$ is obvious, or $I \notin \mathrm{Intv}(C_0)$ and we have that $\mathrm{erep}(I) = p \in J_0(D') \subseteq \mathrm{SRep}(C_0, \mathrm{lab}'_0, D')$. Therefore, (3.5) holds.

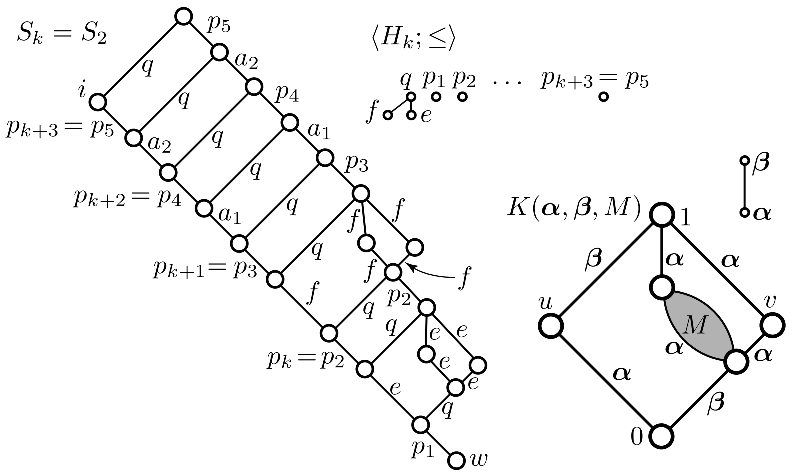

FIGURE 3. S_k for $k = 2$, $\langle H_k; \leq \rangle$, and $K(\boldsymbol{\alpha}, \boldsymbol{\beta}, M)$; the elements $a_1, \dots, a_k$ are defined after (3.2)

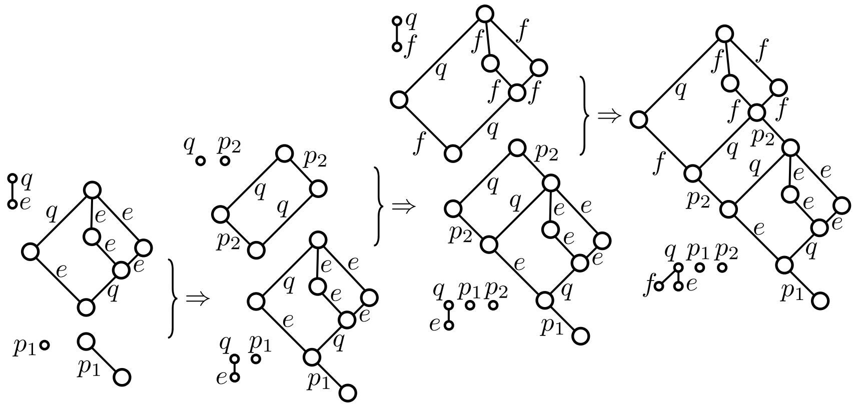

FIGURE 4. Elementary steps towards (5.3)

It is straightforward to check that $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is colored (not only quasi-colored) by the two-element chain $\{\boldsymbol{\alpha} < \boldsymbol{\beta}\}$, as indicated in Figure 1. Of course, we can rename the elements of this chain. For later reference, let M be a simple lattice, and let $K(\boldsymbol{\alpha}, \boldsymbol{\beta}, M)$ denote the colored lattice we obtain from $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ so that we replace its thick prime interval, see Figure 1, by M as indicated in Figure 3; all edges of M are colored by $\boldsymbol{\alpha}$. In Section 7, we will rely on the obvious fact that

> whatever we do with $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ in this section, we could do it with $K(\boldsymbol{\alpha}, \boldsymbol{\beta}, M_1)$, $K(\boldsymbol{\alpha}, \boldsymbol{\beta}, M_2)$, $\dots$, where M_1, M_2,$\dots$ are finite simple lattices. (5.2)

Let S_k be the lattice defined by Figure 3. Also, this figure defines an ordered set $\langle H_k; \leq\rangle$. Let $\widetilde{\gamma}_k\colon \text{Prime}(S_k) \to H_k$ be given by the labeling; we claim that

$$\widetilde{\gamma}_k\colon \text{Prime}(S_k) \to \langle H_k; \leq\rangle \text{ is a coloring; see Figure 3.} \tag{5.3}$$

Note that the *colors*, see (5.3), of the edges of the chain $[w, i]$ of S_k are the same as the *labels*, see (2.4), of the edges of the corresponding filter of C^*. We obtain (5.3) by applying Lemma 4.3 repeatedly; the first three steps are given if Figure 4; the rest of the steps are straightforward. In Figure 4, going from left to right, we construct larger and larger quasi-colored (in fact, colored) lattices by Hall-Dilworth gluing. The colors are given by labeling and their ranges by small diagrams in which the elements are given by half-sized little circles. The action of gluing is indicated by "$\} \Rightarrow$". Now that we have decomposed the task into elementary steps, we can conclude (5.3) easily.

Next, let $\gamma'_{\text{nat}}\colon \text{Prime}(L') \to J(\text{Con}(L'))$ be the natural coloring of L', that is, for $\mathfrak{p} \in \text{Prime}(L')$, we have that $\gamma'_{\text{nat}}(\mathfrak{p}) = \text{con}_{L'}(\mathfrak{p})$. Using that $\varphi'\colon \text{Con}(L') \to D'$ is a lattice isomorphism, see after (3.5), we conclude that its restriction $\psi' := \varphi'\rceil_{J(\text{Con}(L'))}$ is an order isomorphism from $J(\text{Con}(L'))$ onto $J(D')$. Therefore, the composite map $\gamma_1 := \psi' \circ \gamma'_{\text{nat}}$ is a coloring $\gamma_1\colon \text{Prime}(L') \to \langle J(D'); \leq\rangle$. For later reference, note that

$$\psi'(\text{con}_{L'}(\mathfrak{p})) = \psi'(\gamma'_{\text{nat}}(\mathfrak{p})) = (\psi' \circ \gamma'_{\text{nat}})(\mathfrak{p}) = \gamma_1(\mathfrak{p}). \tag{5.4}$$

By 2.1(iiib), C^* is a filter of L', whereby the filter $\uparrow_{L'} w$ is a chain. Hence, L is the Hall–Dilworth gluing of L' and S_k; compare Figures 1 and 3. As a preparation to the next application of Lemma 4.3, we denote the ordered sets $\langle J(D'); \leq\rangle$ and $\langle H_k; \leq\rangle$ also by $\langle J(D'); \nu_1\rangle$ and $\langle H_k; \nu_2\rangle$, respectively. We let $\gamma_2 = \widetilde{\gamma}_k$; see (5.3). Finally, L' and S_k will also be denoted by L_1 and L_2, respectively. With these notations, let γ be the map defined in (4.4). On the set $H := J(D') \cup H_k$, we define a quasiordering ν according to (4.5); note that the pairs $\langle a_1, a_1\rangle, \dots, \langle a_k, a_k\rangle$ required by (4.5) can be omitted. This means that

$$\begin{aligned}\nu = \text{quo}\bigl(\nu_1 \cup \nu_2 \cup \{&\langle p, p_1\rangle, \langle p_1, p\rangle, \langle p, p_2\rangle, \langle p_2, p\rangle, \dots,\\ &\langle p, p_{k+3}\rangle, \langle p_{k+3}, p\rangle\}\bigr).\end{aligned} \tag{5.5}$$

We conclude from Lemma 4.3 that

$$\gamma\colon \text{Prime}(L) \to \langle H; \nu\rangle \text{ is a quasi-coloring.} \tag{5.6}$$

Let $\delta\colon \langle H; \nu\rangle \to \langle J(D); \leq\rangle$ be the map defined by

$$\delta(x) = \begin{cases} p, & \text{if } x \in \{p_1, p_2, \dots, p_{k+3}\},\\ x, & \text{otherwise.}\end{cases} \tag{5.7}$$

Observe that if $\langle x, y\rangle \in \nu_1 \cup \nu_2$ or $x, y \in \{p, p_1, \dots, p_{k+3}\}$, then $\delta(x) \leq \delta(y)$. Hence, the set generating ν in (5.5) is a subset of $\overrightarrow{\text{Ker}}(\delta)$; see (4.2) and thereafter. This implies that $\nu \subseteq \overrightarrow{\text{Ker}}(\delta)$ since $\overrightarrow{\text{Ker}}(\delta)$ is a quasiordering. The inclusion just obtained means that δ is a homomorphism. In order to verify the converse inclusion, $\vec{\mathrm{Ker}}(\delta) \subseteq \nu$, assume that $x \neq y$ and $\langle x, y\rangle \in \vec{\mathrm{Ker}}(\delta)$, that is, $\delta(x) \leq \delta(y)$ in $J(D)$. There are four cases to consider.

First, assume that $x, y \in J(D')$. Then $x = \delta(x) \leq \delta(y) = y$ in $J(D)$. But $J(D')$ is a subposet of $J(D)$, whereby $\langle x, y\rangle \in \nu_1 \subseteq \nu$, as required.

Second, assume that $\{x, y\} \cap J(D') = \emptyset$. Since $x, y \in \{p_1, \dots, p_{k+3}, q\}$, $x \neq y$, $\{\delta(x), \delta(y)\} \subseteq \{p, q\}$, $p \parallel q$, and $\delta(x) \leq \delta(y)$, we conclude that x and y belong to $\{p_1, \dots, p_{k+3}\}$, whereby the required containment $\langle x, y\rangle \in \nu$ is clear by (5.5).

Third, assume that $x \in J(D')$ but $y \notin J(D')$. If $y \in \{p_1, \dots, p_{k+3}\}$, then the required $\langle x, y\rangle \in \nu$ follows from $\langle x, p\rangle \in \nu_1 \subseteq \nu$ and $\langle p, y\rangle \in \nu$. Otherwise, $y = q$, and $x = \delta(x) \leq \delta(q) = q$ gives that $\langle x, e\rangle \in \nu_1 \subseteq \nu$ or $\langle x, f\rangle \in \nu_1 \subseteq \nu$. Since $\langle e, q\rangle, \langle f, q\rangle \in \nu_2 \subseteq \nu$, the required $\langle x, y\rangle \in \nu$ follows by transitivity.

Fourth, assume that $x \notin J(D')$ but $y \in J(D')$. Since $\delta(x) \in \{p, q\}$ and $\delta(x) \leq \delta(y) = y \in J(D')$, the only possibility is that $\delta(x) = p$, x is in $\{p_1, \dots, p_{k+3}\}$, and $y = p$, whereby (5.5) yields the required $\langle x, y\rangle \in \nu$. Therefore, $\vec{\mathrm{Ker}}(\delta) \subseteq \nu$. Thus, it follows from (5.6) and Lemma 4.2 that the map

$$\widehat{\gamma} = \delta \circ \gamma \colon \mathrm{Prime}(L) \to \langle J(D); \leq\rangle, \qquad \text{defined by } \mathfrak{r} \mapsto \delta(\gamma(\mathfrak{r})), \tag{5.8}$$

is a coloring; this coloring is the same what the labeling in Figure 1 suggests. By Lemma 4.1, the map $\mu\colon \langle J(\mathrm{Con}(L)); \leq\rangle \to \langle J(D); \leq\rangle$ described in the lemma is an order isomorphism. By the well-known structure theorem of finite distributive lattices, μ extends to a unique isomorphism $\varphi\colon \mathrm{Con}(L) \to D$. We claim that

$$\text{an element } x \text{ of } D \text{ belongs to } \varphi(\mathrm{Princ}(L)) \text{ if and only if there is a chain } u_0 \prec u_1 \prec \cdots \prec u_n \text{ in } L \text{ such that } x = \widehat{\gamma}([u_0, u_1]) \vee \cdots \vee \widehat{\gamma}([u_{n-1}, u_n]). \tag{5.9}$$

We are going to derive this fact only from the assumption that $\widehat{\gamma}$ is a coloring and φ is the isomorphism what $\widehat{\gamma}$ determines by Lemma 4.1; see between (5.8) and (5.9).

In order to prove (5.9), assume that there is such a chain. Then, applying Lemma 4.1 at the second equality below,

$$\begin{aligned} x &= \widehat{\gamma}([u_0, u_1]) \vee \cdots \vee \widehat{\gamma}([u_{n-1}, u_n]) \\ &= \mu(\mathrm{con}(u_0, u_1)) \vee \cdots \vee \mu(\mathrm{con}(u_{n-1}, u_n)) \\ &= \varphi(\mathrm{con}(u_0, u_1)) \vee \cdots \vee \varphi(\mathrm{con}(u_{n-1}, u_n)) \\ &= \varphi(\mathrm{con}(u_0, u_1) \vee \cdots \vee \mathrm{con}(u_{n-1}, u_n)) = \varphi(\mathrm{con}(u_0, u_n)), \end{aligned} \tag{5.10}$$

which implies that $x \in \varphi(\mathrm{Princ}(L))$. Conversely, assume $x \in \varphi(\mathrm{Princ}(L))$, that is, $x = \varphi(\mathrm{con}(a, b))$ for some $a \leq b \in L$. Let us pick a maximal chain $a = u_0 \prec u_1 \prec \cdots \prec u_n = b$ in the interval $[a, b]$, then (5.10) shows that x is of the required form. This proves the validity of (5.9).

We say that a (5.9)-*chain* $u_0 \prec u_1 \prec \cdots \prec u_n$ *produces* x if the equality in (5.9) holds. In order to show that $Q = \varphi(\mathrm{Princ}(L))$, we need to show that an element $x \in D$ is produced by a (5.9)-chain iff $x \in Q$. It suffices to consider join-reducible elements and chains of length at least two, because chains of length 1 produce join-irreducible elements that are necessarily in Q and, in addition, every join-irreducible x is produced by a (5.9)-chain of length 1 since $\widehat{\gamma}$ is surjective.

First, assume that $x \in Q \setminus J_0(D)$. If $x \in Q \setminus {\downarrow} p$, then x is of the form $x = a_i \vee q$, and we can clearly find a chain of length 2 in $S_k \subseteq L$ with $\widehat{\gamma}$-colors a_i and q, and this is a (5.9)-chain that produces x. Otherwise, $x \in Q' = Q \cap {\downarrow} p$. We obtain from (5.4) that the coloring $\gamma_1 \colon \text{Prime}(L') \to \langle J(D'); \leq \rangle$ determines the order isomorphism $\psi' \colon J(\text{Con}(L') \to \langle J(D'); \leq \rangle$ in the same way as the coloring in Lemma 4.1 determines μ. By the structure theorem of finite distributive lattices, ψ' has exactly one extension to a $\text{Con}(L') \to D'$ isomorphism; this extension is φ' since $\psi' = \varphi' \rceil_{J(\text{Con}(L'))}$; see the paragraph above (5.4). Thus,

$$\begin{array}{l}\gamma_1 \text{ determines } \varphi' \text{ in the same way as } \widehat{\gamma} \\ \text{determines } \varphi\text{; see between (5.8) and (5.9).}\end{array} \tag{5.11}$$

From (4.4), the paragraph preceding (5.4), and (5.6) we see that γ extends γ_1. So, since δ defined in (5.7) acts identically on $J(D')$, $\widehat{\gamma}$ defined in (5.8) also extends γ_1. Combining this fact with (5.11), we obtain that

$$\varphi \text{ and } \widehat{\gamma} \text{ extend } \varphi' \text{ and } \gamma_1 \text{, respectively.} \tag{5.12}$$

By the choice of $\varphi' \colon \text{Con}(L') \to D'$, we have that $x \in Q' = \varphi'(\text{Princ}(L'))$. Therefore, (5.9) applied to $\langle D', L', \varphi', \gamma_1 \rangle$ rather than to $\langle D', L', \varphi', \widehat{\gamma} \rangle$, the sentence after (5.9), (5.11), and (5.12) imply that there exists a (5.9)-chain in D (in fact, even within D') that produces x.

Second, assume that x is produced by a (5.9)-chain W of length at least 2. If W has a p-colored edge with respect to $\widehat{\gamma}$, then ${\uparrow} p \subseteq Q$ implies that $x \in Q$. Hence, with respect to $\widehat{\gamma}$,

$$\text{we can assume that no edge of } W \text{ is colored by } p. \tag{5.13}$$

If W is a chain in L', then applying (5.9) to $\langle D', L', \varphi', \gamma_1 \rangle$ rather than to $\langle D, L, \varphi, \widehat{\gamma} \rangle$ and using (3.6) and (5.12), we obtain that the element x belongs to $\varphi'(\text{Princ}(L')) = Q' \subseteq Q$. If W had an edge both inside L' and outside L', that is, if $\emptyset \neq \text{Prime}(W) \cap \text{Prime}(L') \neq \text{Prime}(W)$, then W would have an edge $[u_{j-1}, u_j]$ such that $u_{j-1} \in L' \setminus C^*$ but $u_j \in C^*$, since L is a Hall-Dilworth gluing of L' and S_k. In Figure 1, $[u_{j-1}, u_j]$ is one of the dashed lines. It would follow from Theorem 2.1(iiid) and (5.12) that $\widehat{\gamma}([u_{j-1}, u_j]) = 1_{D'} = p$, which would contradict (5.13). Thus, W cannot have an edge both inside L' and outside L'. We are left with the case where W is a chain in S_k. In S_k, any two edges of W with distinct $\widehat{\gamma}_k$-colors outside $\{p_1, \dots, p_{k+3}, q\}$ are separated by an edge of W whose $\widehat{\gamma}_k$-color is in $\{p_1, \dots, p_{k+3}, q\}$; see Figure 3. Formulating this within L with $\widehat{\gamma}$ rather than $\gamma_2 = \widehat{\gamma}_k$, any two edges of W with distinct $\widehat{\gamma}$-colors not in $\{p, q\}$ are separated by an edge of W whose $\widehat{\gamma}$-color is in $\{p, q\}$; see Figure 1. Hence, the structure of S_k, see Figures 1 and 3, and (5.13) imply that x is one of the elements $e = e \vee e$, $f = f \vee f$, $e \vee q = q$, $f \vee q = q$, and $a_i \vee q$ for $i = 1, \dots k + 3$, and these elements belong to Q.

Hence, $x \in Q$ for every W. Consequently, $Q = \varphi(\mathrm{Princ}(L))$. This completes the proof of the (iii) ⇒ (i) part of Corollary 1.4. □

6. A new approach to Grätzer's Theorem 2.1

Part 2.1(i) is proved in Grätzer [15]; we do not have anything to add. Part 2.1(ii) is an evident consequence of (the more general but technical) part 2.1(iii). This section is devoted to the proof of part 2.1(iii). Our approach includes a lot of ingredients from Grätzer [15].

Proof of 2.1(iii). Assume that Q is a chain-representable subset of a finite distributive lattice D; see Figure 5, where Q consists of the grey-filled elements. The largest elements of D will be denoted by $\mathbf{1}$, it belongs to $J(D)$ by our assumption. Let $\langle C, \mathrm{lab}, D\rangle$ be a $J(D)$-labeled chain representing Q. We need to find a lattice L and an isomorphism $\varphi\colon \mathrm{Con}(L) \to D$ that satisfy the requirements of 2.1(iii).

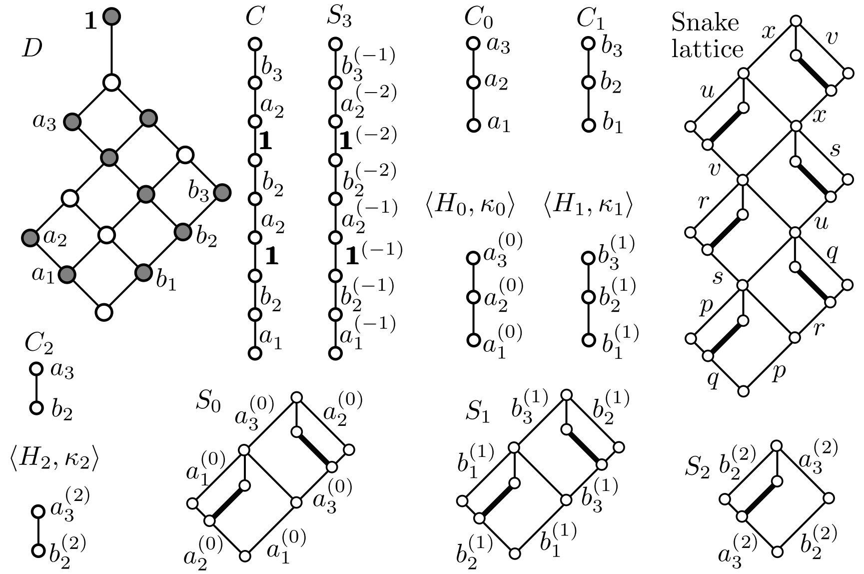

FIGURE 5. An example for $Q \subseteq D$ and the first steps towards its representation

The ordering of $J(D)$ will often be denoted by κ'. Take a list

$$\langle C_0; \kappa'_0\rangle, \ \dots, \ \langle C_{t-1}; \kappa'_{t-1}\rangle$$

of chains in $J(D)$; here κ'_i denotes the restriction of κ' to C_i, for $i < t$. Assume that this list of chains is taken so that $J(D) \setminus \{1_D\} = \bigcup_{i<t} C_i$, and

$$\mathrm{quo}\big((J(D) \times \{\mathbf{1}\}) \cup \bigcup_{i<t} \kappa'_i\big) = \kappa', \text{ that is, } (J(D) \times \{\mathbf{1}\}) \vee \bigvee_{i<t} \kappa'_i = \kappa' \quad (6.1)$$

in $\mathrm{Quo}(J(D))$. Note that although we can always take the list of all chains $\{a,b\}$ with $a \prec_{J(D)} b$, we often get a much smaller lattice L by selecting fewer chains. For example, for the lattice D given in Figure 5, we can let $t=3$, $C_0=\{a_1<a_2<a_3\}$, $C_1=\{b_1<b_2<b_3\}$, and $C_2=\{b_2<a_3\}$. For each of the chains $C_i=\{x_1<x_2<\cdots<x_{m_i}\}=\langle C_i;\kappa_i'\rangle$ such that $m_i>1$, let

$$H_i=\{x_1^{(i)}<x_2^{(i)}<\cdots<x_{m_i}^{(i)}\}=\langle H_i;\kappa_i\rangle \tag{6.2}$$

be an alter ego of C_i; see Figure 5 again. Each of the $\langle H_i;\kappa_i\rangle$, for $i<t$, determines a *snake lattice*, which is obtained by gluing copies of $K(\boldsymbol{\alpha},\boldsymbol{\beta})$ so that there is a coloring σ_i from the set of prime intervals of the snake lattice onto $\langle H_i;\kappa_i\rangle$. For example, if $\langle H_i;\kappa_i\rangle$ had been $\{p<q<r<s<u<v<x\}$, then the snake lattice would have been the one given on the top right of Figure 5. For our example, *this* exemplary snake lattice is not needed; what we need for our D are the S_i and the colorings $\sigma_i\colon \mathrm{Prime}(S_i)\to\langle H_i;\kappa_i\rangle$, indicated by labels in the figure, for $i<t$. (By space considerations, not all edges are labeled.) The purpose of the alter egos is to make our chains H_i pairwise disjoint. If $m_i=|C_i|=1$, then S_i is the two-element lattice and the coloring σ_i is the unique map from the singleton $\mathrm{Prime}(S_i)$ to $\langle H_i;\kappa_i\rangle :=\langle\{x_1^i\};\kappa_i\rangle$, where κ_i is the only ordering on the singleton set H_i.

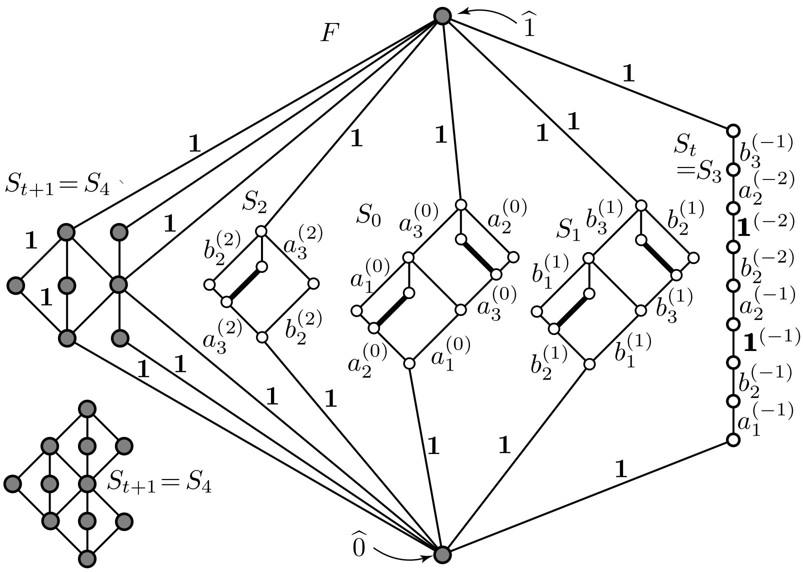

FIGURE 6. The frame lattice F for the example given in Figure 5 and $S_{t+4}=S_4$

Next, we turn $\langle C,\mathrm{lab},D\rangle$, see Figure 5, to a colored lattice S_t as follows. Before its formal definition, note that in our example, $S_t=S_3$ is given in Figure 5. As a lattice, $S_t:=C$. For $x\in J(D)$, let $h(x)=|\{\mathfrak{p}\in\mathrm{Prime}(C):$

$\text{lab}(\mathfrak{p}) = x\}|$; note that $h(x) \geq 1$ since lab is a surjective map. Let

$$H_t := \bigcup_{x \in J(D)} \{x^{(-1)}, \dots, x^{(-h(x))}\} \quad \text{and} \quad \kappa_t := \{\langle y, y\rangle : y \in H_t\}; \tag{6.3}$$

then $\langle H_t; \kappa_t\rangle$ is an antichain, which is not given in the figure. Define the map

$$\sigma_t\colon \text{Prime}(S_t) \to H_t \text{ by the rule } \mathfrak{p} \mapsto x^{(-i)} \text{ iff } \text{lab}(\mathfrak{p}) = x \text{ and, counting from below, } \mathfrak{p} \text{ is the } i\text{-th edge of } C \text{ labeled by } x. \tag{6.4}$$

Less formally, we make the labels of $S_t = C$ pairwise distinct by using negative superscripts; see Figure 5. (The superscripts are negative, because the positive ones have been used for another purpose in (6.2).) The new labels with negative superscripts form an antichain H_t, and the new labeling σ_t becomes a coloring.

The colored lattices S_i, $i \leq t$, with their colorings $\sigma_i\colon \text{Prime}(S_i) \to \langle H_i; \kappa_i\rangle$ will be referred to under the common name *branches*. So the i-th branch is a snake lattice or the two-element lattice for $i < t$, and it is the colored chain S_t with the coloring given in (6.4) for $i = t$.

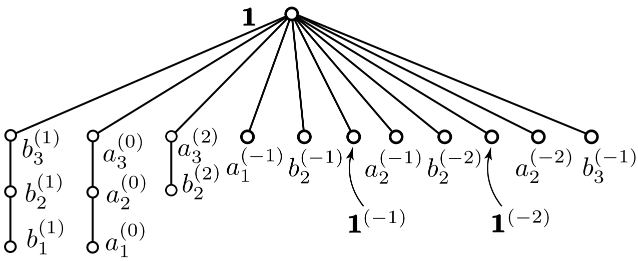

FIGURE 7. $\langle H; \kappa\rangle$

Next, let $\langle H_{t+1}; \kappa_{t+1}\rangle$ be the one-element ordered set $H_{t+1} = \{\mathbf{1}\}$, and

$$\text{let } S_{t+1} \text{ be an arbitrary simple lattice with } |S_{t+1}| \geq 3. \tag{6.5}$$

In Figure 6, S_{t+1} consists of the elements given by a bit larger and grey-filled circles. We have chosen this simple lattice because it is easy to draw. The zero and the unit of S_{t+1} will be denoted by $\widehat{0}$ and $\widehat{1}$, respectively. Note the notational difference: while $\widehat{0}$ and $\widehat{1}$ are in S_{t+1} and they will be in the lattice L we are going to construct, $\mathbf{1}$ and $\mathbf{0}$ are in D and H, and they will correspond to congruences of L. The unique map $\text{Prime}(S_{t+1}) \to \langle H_{t+1}; \kappa_{t+1}\rangle$ will be denoted by σ_{t+1}; it is a coloring. All the lattices and ordered sets mentioned so far in this section are assumed to be pairwise disjoint. Let F be the lattice we obtain from S_{t+1} by inserting all the S_i for $i \leq t$ as intervals such that for every $i, j \in \{0, \dots, t\}$, $x_i \in S_i$, $x_j \in S_j$ and $y_{t+1} \in S_{t+1} \setminus \{\widehat{0}, \widehat{1}\}$, we have that $x_i \vee y_{t+1} = \widehat{1}$, $x_i \wedge y_{t+1} = \widehat{0}$, and if $i \neq j$, then we also have that $x_i \vee x_j = \widehat{1}$ and $x_i \wedge x_j = \widehat{0}$; see Figure 6, which gives F for our example of $Q \subseteq D$ given in Figure 5. Following Grätzer [15], we call F the *frame* or the *frame lattice* associated with $Q \subseteq D$, but note that it depends also on the list of chains and the choice of S_{t+1}. The simplicity of S_{t+1} guarantees that F is a $\{\widehat{0},\widehat{1}\}$-separating lattice. In order to see this, let $x \in L \setminus \{\widehat{0},\widehat{1}\}$. If $x \in S_{t+1}$, then $\mathrm{con}(\widehat{0},x) = \mathrm{con}(x,\widehat{1}) = 1_{\mathrm{Con}(F)}$ by the simplicity of F. If $x \notin S_{t+1}$, then x has a complement y in S_{t+1}, and $\mathrm{con}(\widehat{0},x) = \mathrm{con}(x,\widehat{1}) = 1_{\mathrm{Con}(F)}$ since the same holds for y. Let

$$H := \bigcup_{i<t+2} H_i, \quad \kappa := (H \times \{\mathbf{1}\}) \cup \bigcup_{i<t+2} \kappa_i, \quad \text{and define}$$
$$\sigma(\mathfrak{p}) = \begin{cases} \sigma_i(\mathfrak{p}), & \text{if } \mathfrak{p} \in \mathrm{Prime}(S_0) \cup \dots \cup \mathrm{Prime}(S_t), \\ \mathbf{1}, & \text{otherwise.} \end{cases} \tag{6.6}$$

Clearly, (6.6) above defines a quasiordered set $\langle H;\kappa\rangle$ together with a map $\sigma\colon \mathrm{Prime}(F) \to \langle H;\kappa\rangle$. In case of our example, σ and $\langle H;\kappa\rangle$ are given by Figures 6 and 7, respectively. Clearly, κ is always an ordering. Using that F is $\{\widehat{0},\widehat{1}\}$-separating and arguing similarly to Grätzer [15], it is easy to see that σ is a coloring. For $i < t+1$, the subsets H_i are called the *legs* of H. For $i < t$, the i-th leg H_i is a chain, and it is an antichain for $i = t$.

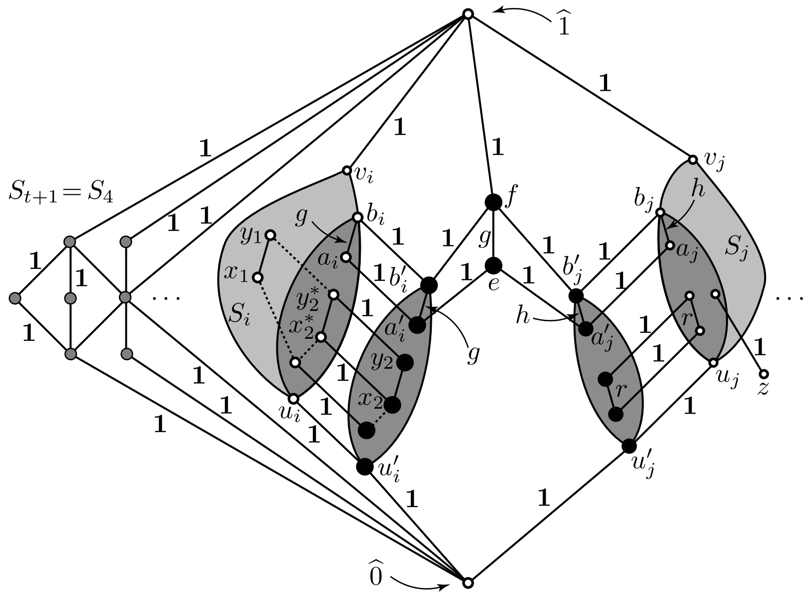

FIGURE 8. Equalizing the colors g and h

Each element of $J(D) \setminus \{\mathbf{1}\}$ has at least one alter ego in H, but generally it has many alter egos; they differ only in their superscripts. The alter egos of an element $x \in J(D)$ with positive superscripts belong to distinct legs. Note at this point that for $x \in J(D)$ and $-1 \geq -s \geq -h(x)$, see (6.3), $x^{(-s)}$ is also called an alter ego of x. Note also that $\mathbf{1}$ also has alter egos, usually many alter egos, in S_t. It is neither necessary, nor forbidden that $\mathbf{1}$ has alter egos in $S_0 \cup \cdots \cup S_{t-1}$. Next, let

$$\begin{aligned}\varepsilon = \{&\langle g_0, h_0\rangle, \langle h_0, g_0\rangle, \langle g_1, h_1\rangle, \langle h_1, g_1\rangle, \dots, \\ &\langle g_{m-1}, h_{m-1}\rangle, \langle h_{m-1}, g_{m-1}\rangle\} \subseteq H^2\end{aligned} \tag{6.7}$$

be a symmetric relation such that the equivalence relation $\mathrm{equ}(\varepsilon)$ generated by ε is the least equivalence on H that collapses every element with all of its alter egos. Every "original color" x (that is, every $x \in J(D)$) has an alter ego in H_t and also in some other leg H_{i_0} such that $i_0 < t$. Quasi-colorings are surjective by definition. Thus, since σ extends σ_i for $i < t+1$ by (6.6) and x has only one alter ego in H_{i_0} for $i_0 < t$, we can assume that

for every $\langle g_\ell, h_\ell\rangle \in \varepsilon$, there are *distinct* branches S_i and S_j such that S_i contains an edge $\mathfrak{p}_i$ with $\sigma(\mathfrak{p}_i) = \sigma_i(\mathfrak{p}_i) = g_\ell$ and S_j contains an edge $\mathfrak{p}_j$ with $\sigma(\mathfrak{p}_j) = \sigma_j(\mathfrak{p}_j) = h_\ell$. (6.8)

Note that the smaller the ε is, the smaller the lattice L will be. Since ε is symmetric,

the equivalence $\mathrm{equ}(\varepsilon)$ generated by ε is $\mathrm{quo}(\varepsilon)$. (6.9)

Hence, letting $\eta := \mathrm{quo}(\kappa \cup \varepsilon)$, we have that $\mathrm{equ}(\varepsilon) \subseteq \eta$. (6.10)

We claim that for every $x, y \in J(D)$,

$x \le y$ in $J(D)$ iff there exist alter egos x' and y' of x and y, respectively, such that $x' \le_\eta y'$. (6.11)

In order to see this, assume first that $x \le y$ in $J(D)$, that is, $x \le_{\kappa'} y$. We can assume that $y \neq \mathbf{1}$, because otherwise $\langle x, y\rangle \in \kappa$ by (6.6), whereby $\langle x, y\rangle \in \eta$; see (6.10). By (6.1), there is a sequence $x = z_0, z_1, \dots, z_n = y$ in $J(D) \setminus \{\mathbf{1}\}$ such that $\langle z_j, z_{j+1}\rangle \in \bigcup_{i<t} \kappa'_i$ for all $j < n$. So for every $j < n$, we can pick an $i_j \in \{0, \dots, t-1\}$ such that $\langle z_j, z_{j+1}\rangle \in \kappa'_{i_j}$, that is, $\langle z_j^{(i_j)}, z_{j+1}^{(i_j)}\rangle \in \kappa_{i_j}$. Hence,

$$\langle z_j^{(i_j)}, z_{j+1}^{(i_j)}\rangle \in \kappa_{i_j} \overset{(6.6)}{\subseteq} \kappa \overset{(6.10)}{\subseteq} \eta \text{ hold for all } j < n. \tag{6.12}$$

For all $j < n-1$, the elements $z_{j+1}^{(i_j)}$ and $z_{j+1}^{(i_{j+1})}$ are alter egos of the same z_{j+1}, whereby (6.9) and (6.10) give that $\langle z_{j+1}^{(i_j)}, z_{j+1}^{(i_{j+1})}\rangle \in \eta$. This fact, (6.12), and the transitivity of η imply that $\langle x', y'\rangle := \langle z_0^{(i_0)}, z_n^{(i_{n-1})}\rangle \in \eta$. That is, there exist alter egos x' and y' of x and y, respectively, such that $x' \le_\eta y'$.

Second, to prove the converse implication, assume that $x' \le_\eta y'$ for alter egos of x and y, respectively. Again, we can assume that $y \neq \mathbf{1}$. It suffices to deal with the particular case $x' \le_{\kappa_i} y'$, because the case $\langle x', y'\rangle \in \varepsilon$ causes no problem and the general case follows from the particular one by (6.6), (6.10), and transitivity. But $x' \le_{\kappa_i} y'$ means that x and y belong to the same chain C_i and $x \le y$ in this chain. Hence, $x \le y$ in $J(D)$, as required. Therefore, (6.11) holds.

Next, we explain where the rest of the proof and that of the construction go. Let $\delta\colon \langle H; \eta\rangle \to \langle J(D); \le\rangle$, defined by $\delta(x) = y$ iff x is an alter ego of y. Since $\overrightarrow{\mathrm{Ker}}(\delta) = \eta$ by (6.11), we conclude that δ is a homomorphism; see between (4.2) and Lemma 4.2. Assume that we can find a lattice L and a map γ such that

$$\gamma\colon \mathrm{Prime}(L) \to \langle H; \eta\rangle \text{ is a coloring.} \tag{6.13}$$

Then, using that $\langle J(D); \leq\rangle$ is an ordered set, not just a quasiordered one, Lemma 4.2 will imply that the map $\widehat{\gamma} := \delta \circ \gamma$ is a coloring $\widehat{\gamma}\colon \mathrm{Prime}(L) \to \langle J(D); \leq\rangle$. In the next step, it will turn out by Lemma 4.1 that

$$\begin{aligned}&\mu\colon \langle J(\mathrm{Con}(L)); \leq\rangle \ \to \ \langle J(D); \leq\rangle, \text{ defined by}\\ &\mathrm{con}(\mathfrak{p}) \mapsto \widehat{\gamma}(\mathfrak{p}), \text{ in an order isomorphism.}\end{aligned} \tag{6.14}$$

Furthermore, (5.9) will be valid for L by the same reason as in Section 5; see the sentence following (5.9). At present, by the choice of S_t and the definition of F,

$$\begin{aligned}&\text{the elements } x \in D \text{ that are of the form described in (5.9),}\\ &\text{with } F \text{ instead of } L, \text{ are exactly the elements of } Q.\end{aligned} \tag{6.15}$$

For $\ell \in \{0, 1, \dots, m\}$, see (6.7), let

$$\varepsilon_\ell := \{\langle g_j, h_j\rangle : j < \ell\} \cup \{\langle h_j, g_j\rangle : j < \ell\} \ \text{ and } \ \eta_\ell := \mathrm{quo}(\kappa \cup \varepsilon_\ell). \tag{6.16}$$

By (6.10), $\eta_0 = \kappa$, $\varepsilon_m = \varepsilon$, and $\eta_m = \eta$. By induction, we are going to find $\{0, 1\}$-separating lattices $L_0 = F$, L_1, ..., L_m and quasi-colorings $\gamma_0 = \sigma\colon \mathrm{Prime}(L_0) \to \langle H; \eta_0\rangle$ and, for $\ell \in \{1, \dots, m\}$, $\gamma_\ell\colon \mathrm{Prime}(L_\ell) \to \langle H; \eta_\ell\rangle$ such that " the elements described in (5.9) remain the same", that is, for every ℓ in $\{1, \dots, m\}$ and $x \in L_\ell$,

$$\begin{aligned}&x = \widehat{\gamma}_\ell([u_0, u_1]) \vee \dots \vee \widehat{\gamma}_\ell([u_{n-1}, u_n]) \text{ for some elements}\\ &u_0 \prec u_1 \prec \dots \prec u_n \text{ of } L_\ell \text{ iff the same equality holds for}\\ &\text{some elements } u_0 \prec u_1 \prec \dots \prec u_n \text{ of } C.\end{aligned} \tag{6.17}$$

Roughly speaking, (6.17) says that (6.15) remains valid. Note that γ_ℓ will extend $\gamma_{\ell-1}$, for $\ell \in \{1, \dots, m\}$. Since $\gamma_0 = \sigma$ and $L_0 := F$ satisfy the requirements, including (6.17), it is sufficient to deal with the transition from $L_{\ell-1}$ to L_ℓ, for $1 \leq \ell \leq m$.

So we assume that $\gamma_{\ell-1}\colon \mathrm{Prime}(L_{\ell-1}) \to \langle H; \eta_{\ell-1}\rangle$ satisfies the requirements formulated in the induction hypothesis above, including (6.17). In order to ease the notation in Figure 8, we denote $\langle g_{\ell-1}, h_{\ell-1}\rangle$ by $\langle g, h\rangle$. Then, as it is clear from (6.16), $\eta_\ell = \mathrm{quo}(\eta_{\ell-1} \cup \{\langle g, h\rangle\} \cup \{\langle h, g\rangle\})$. Hence, we shall add an "equalizing flag" W to $L_{\ell-1}$ such that this flag forces that the congruence generated by a g-colored edge be equal to the congruence generated by an h-colored edge. The term "flag" and its usage are taken from Grätzer [15]. Apart from terminological differences, the argument about our flag is basically the same as that in Grätzer [15]. (Note that our interval $[u_i, b_i]$ in Figure 8 need not be a chain; it is always a chain in Grätzer [15], but this fact is not exploited there.) By (6.8), there are *distinct* $i, j \in \{0, 1, \dots, t\}$ such that we can choose the g-colored edge and the h-colored edge mentioned above from branches S_i and S_j, respectively; see Figure 8. Since the role of g and h is symmetric, we can assume that $i < j$.

In Figure 8, the flag is $[u'_i, b'_i] \cup \{e, f\} \cup [u'_j, b'_j]$. So the large black-filled elements belong to the flag. Note that not all elements of the flag are indicated since neither $[u_i', b_i']$, nor $[u_j', b_j']$ is a chain in general. In order to describe the flag more precisely, let $S_i = [u_i, v_i]$ in $L_{\ell-1}$, and let $[a_i, b_i] \in \text{Prime}(S_i)$ be the interval chosen so that $\gamma_{\ell-1}([a_i, b_i]) = g$. Take the direct product of the dark-grey interval $[u_i, b_i]$ and the two-element chain C_2; this is the dark-grey interval $[u_i', b_i]$ in the figure. Then for every $x \in [u_i, b_i]$, there corresponds a unique element $x' \in [u_i', b_i']$; namely, we obtain x' from x by changing the "C_2-coordinate" of x from 1_{C_2} to 0_{C_2}. We form the Hall–Dilworth gluing of the direct product and $S_i = [u_i, v_i]$ to obtain the interval $[u_i', v_i]$; see Figure 8. In the next step, do exactly the same with j instead of i; see on the right of Figure 8. Finally, add two more elements, e as $a_i' \vee a_j'$ and f as $b_i' \vee b_j'$, as shown in the figure. The lattice we obtain is L_ℓ. Note that Figure 8 contains only a part of L_ℓ; there are more branches in general (indicated by three dots in the figure) and there can be earlier flags with many additional elements; one of these elements is indicated by z on the right of the figure. We extend $\gamma_{\ell-1}$ to a map $\gamma_\ell\colon \text{Prime}(L_\ell) \to \langle H; \eta_\ell \rangle$ as indicated by the figure. In particular, if $[x, y] \in \text{Prime}([u_i, b_i]) \cup \text{Prime}([u_j, b_j])$, then $\gamma_\ell([x', y']) := \gamma_{\ell-1}([x, y])$. We let $\gamma_\ell([e, f]) := g$. (Since $i < j$, $\gamma_\ell([e, f])$ is defined uniquely.) For the rest of the new edges, their γ_ℓ-color is defined to be $\mathbf{1}$. Clearly, L_ℓ satisfies 2.1(iiib) and 2.1(iiid) since so does $L_{\ell-1}$ by the induction hypothesis. By construction, for every chain D of covering elements of L_ℓ, either D has an edge with γ_ℓ-color $\mathbf{1}$, or there is a chain D' in $L_{\ell-1}$ such that

$$\{\gamma_\ell(\mathfrak{r}) : \mathfrak{r} \in \text{Prime}(D)\} = \{\gamma_{\ell_1}(\mathfrak{r}) : \mathfrak{r} \in \text{Prime}(D')\}.$$

Thus, since (6.17) holds for $\ell - 1$ by the induction hypothesis, it holds also for ℓ. Since $\widehat{0}$, $\widehat{1}$, u_i', u_j , and an arbitrarily chosen element of $S_{t+1} \setminus \{\widehat{0}, \widehat{1}\}$ form an M_3 sublattice, $\text{con}(\widehat{0}, u_i') = 1_{\text{Con}(L_\ell)}$, and similarly for $\text{con}(\widehat{0}, u_j')$. Hence, it is easy to see that L_ℓ is a $\{0, 1\}$-separating lattice.

Next, we are going to show that γ_ℓ is a quasi-coloring; the argument runs as follows. Observe that whenever we have a quasi-coloring of a lattice U, then it is straightforward to extend it to $U \times \mathsf{C}_2$: the edges $[x, y]$ and $[x', y']$ have the same color while all the $[x', x]$ edges have the same additional color. Since $L_{\ell-1}$ and L_ℓ are $\{\widehat{0}, \widehat{1}\}$-separating, now the $[x', x]$ edges are $\mathbf{1}$-colored. Apart from e and f, which are so much separated from the rest of L_ℓ that they cannot cause any difficulty, we obtain the flag by two applications of the Hall–Dilworth gluing construction. Hence, the argument given for Lemma 4.3 works here with few and straightforward changes. Only the most important changes and cases are discussed here; namely, the following two.

First, we assume that $[x_1, y_1]$ belongs to $\text{Prime}(L_{\ell-1})$, $[x_2, y_2]$ belongs to $\text{Prime}(L_\ell) \setminus \text{Prime}(L_{\ell-1})$, and $[x_1, y_1] \overset{\text{p-dn}}{\to} [x_2, y_2]$; see Figure 8. We need to show that the inequality $\gamma_\ell([x_1, y_1]) \geq_{\eta_\ell} \gamma_\ell([x_2, y_2])$ holds. The zigzag structure of the flag implies that $\{x_2, y_2\}$ is disjoint from $\{e, f\}$, and it follows that $\{x_2, y_2\} \subseteq [u_i', b_i']$ or $\{x_2, y_2\} \subseteq [u_j', b_j']$. This allows us to assume that $\{x_2, y_2\} \subseteq [u_i', b_i']$. Using that we have a Hall–Dilworth gluing (in the filter $\uparrow u_i'$ of L_ℓ) and $[u_i', b_i] \cong [u_i, b_i] \times \mathsf{C}_2$, we obtain a unique $[x_2^*, y_2^*] \in \text{Prime}([u_i, b_i])$ such that $[x_1, y_1] \overset{\text{p-dn}}{\to} [x_2^*, y_2^*]$, $(x_2^*)' = x_2$ and $(y_2^*)' = y_2$. Since $\gamma_{\ell-1}$ is a quasi-coloring by the induction hypothesis,

$$\gamma_\ell([x_1,y_1]) = \gamma_{\ell-1}([x_1,y_1]) \geq_{\eta_{\ell-1}} \gamma_{\ell-1}([x_2^*,y_2^*]) = \gamma_\ell([x_2,y_2].$$

Since $\eta_{\ell-1} \subseteq \eta_\ell$, this implies the required $\gamma_\ell([x_1,y_1]) \geq_{\eta_\ell} \gamma_\ell([x_2,y_2])$.

Second, assume that $[x_1,y_1]$ and $[x_2,y_2]$ are in $\mathrm{Prime}(L_\ell) \setminus \mathrm{Prime}(L_{\ell-1})$, and $[x_1,y_1] \overset{\text{p-dn}}{\to} [x_2,y_2]$. If none of $\gamma_\ell([x_1,y_1])$ and $\gamma_\ell([x_2,y_2])$ is $\mathbf{1}$, then $\{\gamma_\ell([x_1,y_1]),\gamma_\ell([x_2,y_2])\} \subseteq \{g,h\}$ and $\langle\gamma_\ell([x_1,y_1]),\gamma_\ell([x_2,y_2])\rangle$ belongs to $\mathrm{equ}(\varepsilon_\ell) \subseteq \eta_\ell^{-1}$, as required. Otherwise, both $\gamma_\ell([x_1,y_1])$ and $\gamma_\ell([x_2,y_2])$ are equal to $\mathbf{1}$, and we are ready by reflexivity. This proves 1.4(iii) ⇒ 1.4(i). ☐

7. Taking care of Aut(L)

A lattice M is *automorphism-rigid* if $|\mathrm{Aut}(M)| = 1$. It is well-known from several sources that

$$\text{there exists an infinite set } \{M_0, M_1, M_2, M_3, \dots\} \text{ of pairwise non-isomorphic, at least 3-element, automorphism-rigid, finite lattices;} \tag{7.1}$$

see, for example, Czédli [5, Lemma 2.8], Freese [11] (see also Figure 3 in [20]), Grätzer [17], Grätzer and Quackenbush [20], and Grätzer and Sichler [22] for the validity of this statement or for the construction of a set mentioned in (7.1).

The easiest way to convince ourselves that the lattice L constructed in the preceding sections can be chosen to be automorphism-rigid is to replace the "thick" prime intervals $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3, \dots$ in L by M_1, M_2, $M_3, \dots$ from (7.1), respectively, so that every edge of M_i inherits the color of $\mathfrak{p}_i$; furthermore, based on (6.5), we let $S_{t+1} := M_0$. This is why we have made (5.2) and (6.5) reference points. (Note that instead of using $M_1, M_2, \dots$, there is a more involved way that leads to a smaller lattice L: if an automorphism swaps two distinct snake lattices, then it has to swap two prime intervals of S_t with which these snakes are "equalized", but this is impossible.)

As a particular case of the simultaneous representability of a finite distributive non-singleton lattice D and a finite group G with a finite lattice M in the sense that $D \cong \mathrm{Con}(M)$ and $G \cong \mathrm{Aut}(M)$, it is also known that

$$\text{for every finite group } G, \text{ there exists a finite simple lattice } M_G \text{ such that } G \cong \mathrm{Aut}(M_G) \text{ and } |M_G| \geq 3. \tag{7.2}$$

This simultaneous representability is due to Baranskiĭ [1] and Urquhart [24]; see also Grätzer and Schmidt [21] and Grätzer and Wehrung [23] for even stronger results. Now it is clear how to modify our constructions to complete the proofs.

Completing the proof of Theorem 2.2. First, do the same as in Section 6 but we let $S_{t+1} := M_G$ from (7.2). Then replace the "thick" prime intervals $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3, \dots$ in L by M_1, M_2, $M_3, \dots$ from (7.1), respectively. It follows from (5.2), (6.5), the proof given for the implication 2.1(ii) ⇒ 2.1(iii) in Section 6, and the introductory part of the present section that this method works. ☐

Completing the proof of Theorem 1.3. Assume that D is a finite planar distributive lattice with at least two elements and at most one join-reducible coatom, and let G be a finite group. Also, we can assume that $1_D \notin J(D)$, because otherwise Theorem 2.2 applies.

We are going to follow the construction described in Section 3 and verified in Section 5, but now we shall use Theorem 2.2 rather than Theorem 2.1 to obtain L'. So let $D' = \downarrow p$ as before. Since $|D'| > 1$, we can choose an L' that satisfies the requirements of Theorem 2.2. In particular, $\mathrm{Aut}(L') \cong G$ and $\mathrm{Con}(L') \cong D'$. The construction of L' used some of the lattices listed in (7.1); let i be the smallest subscript such that none of M_i and M_{i+1} was used.

Next, armed with L', construct L as before; see Figure 1. Clearly, each automorphism of L' has a natural extension to an automorphism of L. However, L has two typical automorphisms that we do not want (and so, usually, many others obtained by composition): one of these two automorphisms interchanges the two doubly irreducible elements that are the bottoms of e-colored edges, while the other one does the same with f instead of e. To get rid of these unwanted automorphisms, (5.2) allows us to replace the e-colored thick edge and the f-colored thick edge in Figure 1 by M_i and M_{i+1}, respectively. The new lattice we obtain in this way, which is also denoted by L from now on, has only those automorphisms that are unique extensions of automorphisms of L'. Hence, $\mathrm{Aut}(L) \cong \mathrm{Aut}(L') \cong G$, as required. □

Completing the proof of Corollary 1.4. The implication (iii) ⇒ (ii) is what Theorem 1.3 asserts, while (ii) ⇒ (i) is trivial. Finally, (i) ⇒ (iii) is the same as Czédli [8, Theorem 1.3(ii)⇒(iii)].

In order to present an alternative way to derive that (i) implies (iii), assume that D satisfies (i). By Theorem 2.1(i), D is fully chain-representable. Thus, we obtain from Czédli [8, Proposition 1.6] that D satisfies (iii). □

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