Intersection of the reflexive transitive closures of two rewrite relations induced by term rewriting systems

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Abstract

We show that it is undecidable whether the intersection of the reflexive transitive closures of two rewrite relations induced by term rewriting systems is equal to the reflexive transitive closure of a rewrite relation induced by a term rewriting system.

Keywords: term rewriting systems, Post Correspondence Problem, theory of computation

1 Introduction

In theoretical computer science, in particular in automated theorem proving and term rewriting, a binary relation $\to$ on a set of terms is called a rewrite relation if it is closed both under context application (the "replacement" or "monotonicity" property) and under substitutions (the "fully invariant property"), see Definition 1 in [2] and Definition 4.2.2 in [1]. The inverse, the symmetric closure, the reflexive closure, and the transitive closure of a rewrite relation are again rewrite relations [2]. The intersection of two rewrite relations is again a rewrite relation, and rewrite relations form a complete lattice with respect to intersection, see Section 2.2 in [2]. A term rewrite system (TRS for short) R induces a rewrite relation $\to_R$ [1]. A number of problems to characterise the intersection of various closures of rewrite relations induced by two TRSs have been considered in the literature [4, 6, 7].

By the above discussion, the intersection of the reflexive transitive closures of two rewrite relations induced by TRSs is also a rewrite relation. Hence it is natural to ask whether this intersection is equal to the reflexive transitive closure of a rewrite relation induced by a TRS. We show that the following problems are undecidable:
INSTANCE: Two convergent linear TRSs R and S on the same ranked alphabet Σ.

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QUESTION: Does there exist a TRS U on Σ such that $\rightarrow_R^+ \cap \rightarrow_S^+ = \rightarrow_U^+$?
QUESTION: Does there exist a TRS U on Σ such that $\rightarrow_R^* \cap \rightarrow_S^* = \rightarrow_U^*$?
Here $\rightarrow_R^+$ and $\rightarrow_R^*$ denote the transitive closure and the reflexive transitive closure of $\rightarrow_R$, respectively.

2 Preliminaries

We present a review of the notions, notations and preliminary results used in the paper.

2.1 Abstract Reduction Systems

An abstract reduction system is a pair $(A, \rightarrow)$, where the reduction $\rightarrow$ is a binary relation on the set A. $\leftarrow$, $\leftrightarrow$, $\rightarrow^*$, and $\leftrightarrow^*$ denote the inverse, the symmetric closure, the reflexive transitive closure, and the reflexive transitive symmetric closure of the binary relation $\rightarrow$, respectively. $x \in A$ is irreducible if there is no y such that $x \rightarrow y$. $y \in A$ is a normal form of $x \in A$ if $x \rightarrow^* y$ and y is irreducible. If $x \in A$ has a unique normal form, then it is denoted by $x\downarrow$. $y \in A$ is a descendant of $x \in A$ if $x \rightarrow^* y$.

The reduction $\rightarrow$ is called

- confluent if for all $x, y_1, y_2 \in A$, if $y_1 \leftarrow^* x \rightarrow^* y_2$, then $y_1 \rightarrow^* z \leftarrow^* y_2$ for some $z \in A$;
- terminating if there is no infinite chain $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$;
- convergent if it is both confluent and terminating.

2.2 Terms

$\mathbb{N}$ stands for the set of nonnegative integers, and $[1, n]$ stand for the set $\{1, \ldots, n\}$ for each $n \in \mathbb{N}$. A ranked alphabet is a finite set Σ in which every symbol has a unique rank in $\mathbb{N}$. For each $m \in \mathbb{N}$, Σ_m denotes the set of all elements of Σ which have rank m. The elements of Σ_0 are called constants.

For a set of variables Y and a ranked alphabet Σ, $T_\Sigma(Y)$ denotes the set of Σ-terms (or Σ-trees) over Y, and $id_{T_\Sigma(Y)}$ denotes the identity relation on $T_\Sigma(Y)$. $T_\Sigma(\emptyset)$ is written as T_Σ. A term $t \in T_\Sigma$ is called a ground term. A term $t \in T_\Sigma(Y)$ is linear if each variable in Y occurs at most once in t. We specify a countable set $X = \{x_1, x_2, \ldots\}$ of variables which will be kept fixed in this paper. Moreover, we put $X_m = \{x_1, \ldots, x_m\}$ for $m \in \mathbb{N}$. Hence $X_0 = \emptyset$.

For a term $t \in T_\Sigma(X)$, the height $height(t)$ and the set of positions $POS(t) \subseteq N^*$ of t are defined by tree induction.

- If $t \in \Sigma_0 \cup X$, then $height(t) = 0$ and $POS(t) = \{\lambda\}$.
- If $t = f(t_1, \ldots, t_m)$ with $f \in \Sigma_m,\ m > 0$, then
  $height(t) = 1 + \max\{height(t_i) \mid 1 \le i \le m\}$ and
  $POS(t) = \{ i\alpha \mid 1 \le i \le m, \alpha \in POS(t_i) \}$.

For each $t \in T_\Sigma(X)$ and $\alpha \in POS(t)$, we introduce the subterm $t/\alpha \in T_\Sigma(X)$ of t at α and define the label $lab(t, \alpha) \in \Sigma \cup X$ in t at α as follows:

- for $t \in \Sigma_0 \cup X$, $t/\lambda = t$ and $lab(t, \lambda) = t$;
- for $t = f(t_1, \dots, t_m)$ with $m \geq 1$ and $f \in \Sigma_m$, if $\alpha = \lambda$ then $t/\alpha = t$ and $lab(t, \alpha) = f$, otherwise, if $\alpha = i\beta$ with $i \in [1, m]$, then $t/\alpha = t_i/\beta$ and $lab(t, \alpha) = lab(t_i, \beta)$.

For $t \in T_\Sigma$, $\alpha \in POS(t)$, and $r \in T_\Sigma$, we define $t[\alpha \leftarrow r] \in T_\Sigma$ as follows.

- If $\alpha = \lambda$, then $t[\alpha \leftarrow r] = r$.
- If $\alpha = i\beta$, for some $i \in N$ and $\beta \in N^*$, then $t = f(t_1, \dots, t_m)$ with $f \in \Sigma_m$ and $i \in [1, m]$. Then $t[\alpha \leftarrow r] = f(t_1, \dots, t_{i-1}, t_i[\beta \leftarrow r], t_{i+1}, \dots, t_m)$.

For a term $t \in T_\Sigma(X)$, $var(t)$ denotes the set of variables occurring in t, i.e. $var(t) = \{\, x_i \mid \text{ there exists } \alpha \in POS(t) \text{ such that } lab(t, \alpha) = x_i \,\}$.

For trees $t \in T_\Sigma(X_m)$, and $t_1, \dots, t_m \in T_\Sigma(X)$, we denote by $t[t_1, \dots, t_m]$ the tree obtained by substituting t_i for every occurrence of x_i in t, for each $i \in [1, m]$. We say that $t \in T_\Sigma(X_m)$ is a pattern of $s \in T_\Sigma(X)$, if there are $t_1, \dots, t_m \in T_\Sigma(X)$ such that $s = t[t_1, \dots, t_m]$.

For each $n \in \mathbb{N}$, an n-context over Σ is a term $u \in T_\Sigma(X_n)$ with exactly one occurrence of the variable x_i for each $i \in [1, n]$. $C_{\Sigma,n}$ denotes the set of n-contexts over Σ. Note that $C_{\Sigma,0} = T_\Sigma$. Let $C_\Sigma = \bigcup_{n \in \mathbb{N}} C_{\Sigma,n}$. We call a mapping $\omega : \Sigma \to \mathbb{N}$ a weight function. The weight function ω can be extended to a function $\omega : C_\Sigma \to \mathbb{N}$ as follows: let $\omega(u) = \sum_{f \in \Sigma} \omega(f) \cdot |u|_f$, where $|u|_f$ denotes the number of occurrences of symbol f in u. Thus, $\omega(u)$ simply adds up the weight of all occurrences of symbols of Σ in u.

An alphabet Δ is any finite nonempty set, Δ^* stands for the set of words over Δ, and λ denotes the empty word. For an alphabet Δ, we consider the ranked alphabet $\Delta \cup \{\, \# \,\}$, where $\# \notin \Delta$. Here each element of Δ is a unary symbol and $\#$ is a constant. Then we consider a tree in $T_{\Delta \cup \{\, \# \,\}}$ as a word over the alphabet $\Delta \cup \#$. For example, let $\Delta = \{\, a, b \,\}$. Then the tree $a(b(b(a(\#))))$ is written as the word $abba\#$. Conversely, for each word $w \in \Delta^*$, the word $w\#$ over the alphabet $\Delta \cup \{\, \# \,\}$ can be considered as a tree over the ranked alphabet $\Delta \cup \{\, \# \,\}$. For example, the word $aab\#$ can be considered as the tree $a(a(b(\#)))$.

For an alphabet Δ, we also consider the alphabets $\overline{\Delta} = \{\, \overline{a} \mid a \in \Delta \,\}$ and $\underline{\Delta} = \{\, \underline{a} \mid a \in \Delta \,\}$. The alphabets Δ, $\overline{\Delta}$, and $\underline{\Delta}$ are pairwise disjoint. For each word $w \in \Delta^*$, the word $\overline{w} \in \overline{\Delta}^*$ is defined as follows.

- If $w = \lambda$, then $\overline{w} = \lambda$.
- If $w = az$ for some $a \in \Sigma$ and $z \in \Delta^*$, then $\overline{w} = \overline{a}\,\overline{z}$.

For each word $w \in \Delta^*$, we define the word $\underline{w} \in \underline{\Delta}^*$ in a similar way to $\overline{w}$.

2.3 Term Rewriting Systems

Let Σ be a ranked alphabet. Then a term rewriting system (TRS) R on Σ is a finite subset of $(T_\Sigma(X) - X) \times T_\Sigma(X)$. For an element (l, r) of a TRS R, $var(r)$ is a subset of $var(l)$, and $l \notin X$. Elements (l, r) of R are called rules and are denoted by $l \to r$. The TRS R is linear if for each rule $l \to r$, both l and r are linear. A TRS R is context replacing if for each rule $l \to r$ of R, l and r are n-contexts for some $n \in \mathbb{N}$.

Let R be a TRS over Σ. For any terms $s, t \in T_\Sigma(X)$, position $\alpha \in POS(s)$, and rule $l \to r$ in R with $l, r \in T_\Sigma(X_m)$, $m \in \mathbb{N}$, we say that s rewrites to t applying the rule $l \to r$ at α, and denote this by $s \to_{l \to r, \alpha} t$ if there are $s_1, \ldots, s_m \in T_\Sigma(X)$ such that $s/\alpha = l[s_1, \ldots, s_m]$ and $t = s[\alpha \leftarrow r[s_1, \ldots, s_m]]$. Here we also say that s rewrites to t and denote this by $s \to_R t$.

We say that a TRS R is confluent, terminating, or convergent, if $\to_R$ has the corresponding property. For a term $t \in T_\Sigma$, $R^*(t) = \{ p \mid t \to_R^* p \}$ is the set of descendants of t, and $R^+(t) = \{ p \mid t \to_R^+ p \}$ is the set of proper descendants of t.

Proposition 2.1 *Let TRS R be a context replacing TRS such that for each rule $l \to r$ in R, $\omega(l) > \omega(r)$. Then R is terminating.*

Proof. By direct inspection of the definitions. □

For the concept of a critical pair, see [1] or [5].

Proposition 2.2 *[1, 5]. If a TRS R is terminating and has no critical pairs, then R is convergent.*

2.4 Post Correspondence Problem

A Post Correspondence System (PCS for short) over an alphabet Δ is a pair $\langle \mathbf{w}, \mathbf{z} \rangle = \langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle$, $n \geq 1$, of lists of nonempty words over the alphabet Δ. We say that the nonempty word $i_1 \ldots i_\ell$ over the alphabet $[1, n]$ is a solution of the PCS $\langle \mathbf{w}, \mathbf{z} \rangle$, if

$$w_{i_1} \ldots w_{i_\ell} = z_{i_1} \ldots z_{i_\ell} .$$

cf. [3]. The Post Correspondence Problem is the question whether or not a given PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ has a solution.

Proposition 2.3 [3] *The Post Correspondence Problem is unsolvable. That is, there is no algorithm which takes a PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ as input and determines whether or not there is a solution of the PCS $\langle \mathbf{w}, \mathbf{z} \rangle$.*

3 Main Results

We assign two TRSs R and S to a PCS $\langle \mathbf{w}, \mathbf{z} \rangle$, and study the connection between a solution of $\langle \mathbf{w}, \mathbf{z} \rangle$ and the intersection $\to_R^+ \cap \to_S^+$.

Let $\langle \mathbf{w}, \mathbf{z} \rangle = \langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle$ be a Post Correspondence System over the alphabet Δ. We associate the ranked alphabet Σ and the TRSs R and S over Σ with the PCS $\langle \mathbf{w}, \mathbf{z} \rangle$.

Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, $\Sigma_0 = \{\,\#\,\}$, $\Sigma_1 = \Delta \cup \overline{\Delta} \cup \underline{\Delta} \cup [1,n] \cup \overline{[1,n]} \cup \underline{[1,n]}$, $\Sigma_2 = \{\, f \,\}$. Without loss of generality we may assume that $\Delta \cup \overline{\Delta} \cup \underline{\Delta}$ is disjoint with $[1,n] \cup \overline{[1,n]} \cup \underline{[1,n]}$.

The TRS R consists of the rules

(a) $k\# \to \overline{k}\#$ for each $k \in [1,n]$,

(b) $\delta\# \to \overline{\delta}\#$ for each $\delta \in \Delta$,

(c) $j\overline{k}x_1 \to \overline{j}\overline{k}x_1$ for all $j,k \in [1,n]$,

(d) $\gamma\overline{\delta}x_1 \to \overline{\gamma}\overline{\delta}x_1$ for all $\gamma,\delta \in \Delta$, and

(e) $f(\overline{w_k}x_1, \overline{k}x_2) \to f(x_1,x_2)$ for each $k \in [1,n]$.

The left-hand sides of the rules of type (a) or (b) of R contain symbols in the set $\{\,\#\,\} \cup [1,n] \cup \Delta$. The right-hand sides of the rules of type (a) or (b) of R contain symbols in $\overline{[1,n]} \cup \overline{\Delta}$. Both sides of the rules of type (c) or (d) of R contain a symbol in $\overline{[1,n]} \cup \overline{\Delta}$. The left-hand sides of the rules of type (e) of R contain a symbol in $\overline{[1,n]} \cup \overline{\Delta}$, and applying a rule of type (e) of R, we delete symbols in $\overline{[1,n]} \cup \overline{\Delta}$.

Symmetrically, the TRS S consists of the rules

(a) $k\# \to \underline{k}\#$ for each $k \in [1,n]$,

(b) $\delta\# \to \underline{\delta}\#$ for each $\delta \in \Delta$,

(c) $j\underline{k}x_1 \to \underline{j}\underline{k}x_1$ for all $j,k \in [1,n]$,

(d) $\gamma\underline{\delta}x_1 \to \underline{\gamma}\underline{\delta}x_1$ for all $\gamma,\delta \in \Delta$, and

(e) $f(\underline{z_k}x_1, \underline{k}x_2) \to f(x_1,x_2)$ for each $k \in [1,n]$.

We now define the weight function $\omega : \Sigma \to \mathbb{N}$.

- $\omega(f) = 0$,
- $\omega(g) = 2$, $\omega(\overline{g}) = 1$, and $\omega(\underline{g}) = 1$ for every $g \in \Delta$, and
- $\omega(k) = 2$, $\omega(\overline{k}) = 1$, and $\omega(\underline{k}) = 1$ for every $k \in [1,n]$.

Statement 3.1 *R and S are convergent linear TRSs.*

Proof. Observe that R and S are context replacing TRSs and hence are linear. Furthermore, for each rule $l \to r$ in R, $\omega(l) > \omega(r)$. Consequently, by Proposition 2.1, R is terminating. By direct inspection of the rules of R, we get that R has no critical pairs. Thus, by Proposition 2.2, R is convergent. Analogously, we obtain that S is convergent and linear as well. $\square$

Statement 3.2 *Let the nonempty word $i_1 \ldots i_c$ over the alphabet $[1,n]$ be a solution of the PCS $\langle \mathbf{w}, \mathbf{z} \rangle$. Then*
$R^+(f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1 i_2 \ldots i_c\#)) \cap S^+(f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1 i_2 \ldots i_c\#)) = \{\, f(\#,\#) \,\}$
and
$R^*(f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1 i_2 \ldots i_c\#)) \cap S^*(f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1 i_2 \ldots i_c\#)) = \{\, f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1 i_2 \ldots i_c\#), f(\#,\#) \,\}$.

Proof. Consider the reduction sequence
$f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1 i_2 \ldots i_c\#) \to_R f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1 i_2 \ldots \overline{i_c}\#) \to_R \cdots$
$\to_R f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, \overline{i_1 i_2 \ldots i_c}\#) \to_R \cdots$
$\to_R f(w_{i_1}w_{i_2}\ldots \overline{w_{i_c}}\#, \overline{i_1 i_2 \ldots i_c}\#) \to_R \cdots$
$\to_R f(\overline{w_{i_1}w_{i_2}\ldots w_{i_c}}\#, \overline{i_1 i_2 \ldots i_c}\#) \to_R f(\overline{w_{i_2}\ldots w_{i_c}}\#, \overline{i_2 \ldots i_c}\#) \to_R$
$f(\overline{w_{i_3}\ldots w_{i_c}}\#, \overline{i_3 \ldots i_c}\#) \to_R \cdots \to_R f(\overline{w_{i_c}}\#, \overline{i_c}\#) \to_R f(\#,\#)$.

Here R applies a rule of type (a) in the first step, then rules of type (c), a rule of type (b), then rules of type (d), and finally rules of type (e).

Analogously, we have the reduction sequence

$f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1i_2\ldots i_c\#) \to_S f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1i_2\ldots \underline{i_c}\#) \to_S \cdots$
$\to_S f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, \underline{i_1i_2\ldots i_c}\#) \to_S \cdots$
$\to_S f(w_{i_1}w_{i_2}\ldots \underline{w_{i_c}}\#, \underline{i_1i_2\ldots i_c}\#) \to_S \cdots$
$\to_S f(\underline{w_{i_1}w_{i_2}\ldots w_{i_c}}\#, \underline{i_1i_2\ldots i_c}\#) \to_S f(\underline{w_{i_2}\ldots w_{i_c}}\#, \underline{i_2\ldots i_c}\#) \to_S$
$f(\underline{w_{i_3}\ldots w_{i_c}}\#, \underline{i_3\ldots i_c}\#) \to_S \cdots \to_S f(\underline{w_{i_c}}\#, \underline{i_c}\#) \to_S f(\#, \#)$.

Observe that the symbols in the set $\overline{[1,n]} \cup \overline{\Delta}$ do not appear in the rules of S. Similarly, the symbols in the set $\underline{[1,n]} \cup \underline{\Delta}$ do not appear in the rules of R. Hence

$R^+(f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1i_2\ldots i_c\#)) \cap S^+(f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1i_2\ldots i_c\#)) = \{ f(\#,\#) \}$

and

$R^*(f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1i_2\ldots i_c\#)) \cap S^*(f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1i_2\ldots i_c\#)) = \{ f(w_{i_1}w_{i_2}\ldots w_{i_c}\#, i_1i_2\ldots i_c\#), f(\#,\#) \}$. □

Statement 3.3 *If PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ has a solution, then there is no TRS U on Σ such that $\to_R^* \cap \to_S^* = \to_U^*$.*

Proof. On the contrary, assume that PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ has a solution, and that there is a TRS U on Σ such that $\to_R^* \cap \to_S^* = \to_U^*$. Let $a = max\{ height(l) \mid l \to r \in U \}$. Let the nonempty word $i_1 \ldots i_\ell$ over the alphabet $[1,n]$ be a solution of the PCS $\langle \mathbf{w}, \mathbf{z} \rangle$. Then $b \cdot \ell > a$ for some $b \geq 1$. Furthermore, $(i_1 \ldots i_\ell)^b$ is also a solution of the PCS $\langle \mathbf{w}, \mathbf{z} \rangle$. Let us denote the word $(i_1 \ldots i_\ell)^b$ by $k_1 \ldots k_c$, where $c = b \cdot \ell$, $k_1 = i_1, \ldots, k_\ell = i_\ell, k_{\ell+1} = i_1, \ldots, k_c = i_\ell$. Consequently, by Statement 3.2, $R^*(f(w_{k_1}w_{k_2}\ldots w_{k_c}\#, k_1k_2\ldots k_c\#)) \cap S^*(f(w_{k_1}w_{k_2}\ldots w_{k_c}\#, k_1k_2\ldots k_c\#)) = \{ f(w_{k_1}w_{k_2}\ldots w_{k_c}\#, k_1k_2\ldots k_c\#), f(\#,\#) \}$. Then $U^*(f(w_{k_1}w_{k_2}\ldots w_{k_c}\#, k_1k_2\ldots k_c\#)) = \{ f(w_{k_1}w_{k_2}\ldots w_{k_c}\#, k_1k_2\ldots k_c\#), f(\#,\#) \}$. Hence we have $f(w_1w_2\ldots w_{k_c}\#, k_1k_2\ldots k_c\#) = s_0 \to_U s_1 \to_U s_2 \to_U \cdots \to_U s_d = f(\#,\#)$ for some $d \geq 1$, where $s_1 \neq f(w_{k_1}w_{k_2}\ldots w_{k_c}\#, k_1k_2\ldots k_c\#)$. We now show that $s_1 \neq f(\#,\#)$. On the contrary, assume that $s_1 = f(\#,\#)$. Then along the reduction $f(w_1w_2\ldots w_{k_c}\#, k_1k_2\ldots k_c\#) \to_U f(\#,\#)$, U applies a rule $f(v_1, v_2) \to f(\#,\#)$ at the position λ, where $v_1 \in T_\Sigma(X)$ is a pattern of $w_1w_2\ldots w_{k_c}\#$ and $v_2 \in T_\Sigma(X)$ is a pattern of $k_1k_2\ldots k_c\#$. By the definition of a, and the inequality $c > a$, $v_1 \in T_\Sigma(X)$ is a pattern of $w_1w_2\ldots w_{k_c}1\#$ as well. Thus, $f(w_1w_2\ldots w_{k_c}1\#, k_1k_2\ldots k_c\#) \to_U f(\#,\#)$. However, $f(w_1w_2\ldots w_{k_c}1\#, k_1k_2\ldots k_c\#) \to_R^* f(\#,\#)$ does not hold by $(\Delta \cup \overline{\Delta}) \cap ([1,n] \cup \overline{[1,n]}) = \emptyset$. This contradicts the inclusion $\to_U^* \subseteq \to_R^*$. Thus $s_1 \neq f(\#,\#)$. Therefore $s_1 \notin \{ f(w_{k_1}w_{k_2}\ldots w_{k_c}\#, k_1k_2\ldots k_c\#), f(\#,\#) \}$, which contradicts the equation $U^*(f(w_{k_1}w_{k_2}\ldots w_{k_c}\#, k_1k_2\ldots k_c\#)) = \{ f(w_{k_1}w_{k_2}\ldots w_{k_c}\#, k_1k_2\ldots k_c\#), f(\#,\#) \}$, because $s_1 \in U^+(f(w_1w_2\ldots w_{k_c}\#, k_1k_2\ldots k_c\#))$.

□

Analogously to Statement 3.3, we can show the following.

Statement 3.4 *If PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ has a solution, then there is no TRS U on Σ such that $\to_R^+ \cap \to_S^+ = \to_U^+$.*

Statement 3.5 *Let $s, t \in T_\Sigma(X)$ such that $s \to_R^+ t$ and $s \to_S^+ t$. Then PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ has a solution.*

Proof. Consider the R-reduction sequence (1) and the S-reduction sequence (2).

$$s = p_0 \underset{l_0 \to r_0, \alpha_0}{\to} p_1 \underset{l_1 \to r_1, \alpha_1}{\to} \cdots \underset{l_{m-1} \to r_{m-1}, \alpha_{m-1}}{\to} p_m = t,\ m \geq 1, \tag{1}$$

$$s = q_0 \underset{l'_0 \to r'_0, \beta_0}{\to} q_1 \underset{l'_1 \to r'_1, \beta_1}{\to} \cdots \underset{l'_{d-1} \to r'_{d-1}, \beta_{d-1}}{\to} q_d = t,\ d \geq 1. \tag{2}$$

Claim 3.6 *For each $0 \leq i \leq m-1$ and for each position $\gamma \in POS(p_i)$, if $lab(p_i, \gamma) = f$, then α_i is not a proper prefix of γ.*

Proof. We proceed by induction on i.

Base Case: $i = 0$. Let position $\gamma \in POS(p_0)$ be such that $lab(p_0, \gamma) = f$. We proceed by contradiction. Assume that α_0 is a proper prefix of γ. Observe that the constant # appears in the left-hand sides of rules of types (a) and (b), thus $l_0 \to r_0$ is not of type (a) or (b). The left-hand sides of the rules of type (c) or (d) or (e) of R contain a symbol in $\overline{[1, n]} \cup \overline{\Delta}$. Consequently, if the rule $l_0 \to r_0$ is of type (c) or (d) or (e), then those symbols in $\overline{[1, n]} \cup \overline{\Delta}$ which appear in l_0, also appear in p_0. Recall that the symbols in $\overline{[1, n]} \cup \overline{\Delta}$ do not appear in the rules of S. Hence S cannot simulate the application of $l_0 \to r_0$ along (2). Thus $l_0 \to r_0$ is not of type (c) or (d) or (e) either.

Induction Step: Let $1 \leq i \leq m-1$, and assume that the claim holds for $1, \ldots, i-1$. We proceed by contradiction, and assume that α_i is a proper prefix of γ. Observe that the constant # appears in the left-hand sides of rules of types (a) and (b), thus $l_i \to r_i$ is not of type (a) or (b). The left-hand sides of the rules of type (c) or (d) or (e) of R contain a symbol in $\overline{[1, n]} \cup \overline{\Delta}$. Consequently, if the rule $l_i \to r_i$ is of type (c) or (d) or (e), then those symbols in $\overline{[1, n]} \cup \overline{\Delta}$ which appear in l_i, also appear in p_i. By the induction hypothesis, $lab(p_0, \gamma) = f$, and these symbols also appear in p_0 at the same position as in p_i. Recall that the symbols in the set $\overline{[1, n]} \cup \overline{\Delta}$ do not appear in the rules of S. Hence S cannot simulate the application of $l_i \to r_i$ along (2). Thus $l_i \to r_i$ is not of type (c) or (d) or (e) either. □

Similarly to Claim 3.6, we have the following result. For each $0 \leq i \leq d-1$ and for each position $\gamma \in POS(q_i)$, if $lab(q_i, \gamma) = f$, then α_i is not a proper prefix of γ. Hence there is $\alpha \in POS(s) \cap POS(t)$ such that

- $s/\alpha \to_R^+ t/\alpha$ and $s/\alpha \to_S^+ t/\alpha$,
- $s/\alpha = f(u_1, u_2)$, $u_1, u_2 \in T_\Sigma$, f does not appear in u_1 or u_2, and
- $t/\alpha = f(v_1, v_2)$, $v_1, v_2 \in T_\Sigma$, f does not appear in v_1 or v_2.

In the light of the description of the rules of R presented right after the the definition of R, we argue as follows. By the reductions $s/\alpha \to_R^+ t/\alpha$ and $s/\alpha \to_S^+ t/\alpha$, we have $u_1 = v\#$ and $u_2 = p\#$ for some $v \in \Delta^*$ and $p \in [1, n]^*$. Moreover,

$$s/\alpha = f(v\#, p\#) = \eta_0 \underset{R}{\to} \eta_1 \underset{R}{\to} \cdots \underset{R}{\to} \eta_a = t/\alpha,\ a \geq 1, \tag{3}$$

where R applies a rule of type (a) in the first step, then rules of type (c), a rule of type (b), then rules of type (d), and finally rules of type (e). Consequently, $t/\alpha = f(\#, \#)$, $v = w_{i_1} \dots w_{i_\ell}$ and $p = i_1 \dots i_\ell$ for some nonempty word $i_1 \dots i_\ell$ over the alphabet $[1, n]$.

Symmetrically,

$$s/\alpha = f(v\#, p\#) = \xi_0 \underset{S}{\rightarrow} \xi_1 \underset{S}{\rightarrow} \cdots \underset{S}{\rightarrow} \xi_b = t/\alpha,\ b \geq 1, \tag{4}$$

where S applies a rule of type (a) in the first step, then rules of type (c), a rule of type (b), then rules of type (d), and finally rules of type (e). Therefore, $v = z_{i_1} \dots z_{i_\ell}$. Hence the word $i_1 \dots i_\ell$ is a solution of the PCS $\langle \mathbf{w}, \mathbf{z} \rangle$. □

Corollary 3.7 *If PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ has no solution, then $\rightarrow_R^+ \cap \rightarrow_S^+ = \rightarrow_\emptyset^+$ and $\rightarrow_R^* \cap \rightarrow_S^* = \rightarrow_\emptyset^*$.*

Proof. By contradiction. Assume that $\rightarrow_R^+ \cap \rightarrow_S^+ \neq \rightarrow_\emptyset^+$. Since $\rightarrow_\emptyset^+ = \emptyset$, there are $s, t \in T_\Sigma(X)$ such that $s \rightarrow_R^+ t$ and $s \rightarrow_S^+ t$. Consequently, by Statement 3.5, PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ has a solution.

Assume that $\rightarrow_R^* \cap \rightarrow_S^* \neq \rightarrow_\emptyset^*$. Since $\rightarrow_\emptyset^* = id_{T_\Sigma(X)}$, there are $s, t \in T_\Sigma(X)$ such that $s \rightarrow_R^* t$ and $s \rightarrow_S^* t$ and $s \neq t$. Hence $s \rightarrow_R^+ t$ and $s \rightarrow_S^+ t$. Then, as we have already seen above, PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ has a solution. □

Statements 3.3 and 3.4 and Corollary 3.7 imply the following.

Corollary 3.8 *The following three statements are equivalent.*

- *The PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ has no solution.*
- *There exists a TRS U on Σ such that $\rightarrow_R^+ \cap \rightarrow_S^+ = \rightarrow_U^+$.*
- *There exists a TRS U on Σ such that $\rightarrow_R^* \cap \rightarrow_S^* = \rightarrow_U^*$.*

By Proposition 2.3, Statement 3.1, and Corollary 3.8, we have our main result.

Theorem 3.9 *The following problems are undecidable:*
INSTANCE: Two convergent linear TRSs R and S on the same ranked alphabet Σ.
QUESTION: Does there exist a TRS U on Σ such that $\rightarrow_R^+ \cap \rightarrow_S^+ = \rightarrow_U^+$?
QUESTION: Does there exist a TRS U on Σ such that $\rightarrow_R^* \cap \rightarrow_S^* = \rightarrow_U^*$?

Finally, we present our conjecture.

Conjecture 3.10 *The following problem is undecidable:*
INSTANCE: Two TRSs R and S on the same ranked alphabet Σ.
QUESTION: Does there exist a TRS U on Σ such that $\leftrightarrow_R^* \cap \leftrightarrow_S^* = \leftrightarrow_U^*$?

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