Bernstein- and Markov-type inequalities for rational functions

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Abstract

Asymptotically sharp Bernstein- and Markov-type inequalities are established for rational functions on C^2 smooth Jordan curves and arcs. The results are formulated in terms of the normal derivatives of certain Green's functions with poles at the poles of the rational functions in question. As a special case (when all the poles are at infinity) the corresponding results for polynomials are recaptured.

Contents

1 Introduction

Inequalities for polynomials have a rich history and numerous applications in different branches of mathematics, in particular in approximation theory (see, for example, the books [3], [5] and [15], as well as the extensive references there). The two most classical results are the Bernstein inequality [2]

$$|P_n'(x)| \le \frac{n}{\sqrt{1-x^2}}\|P_n\|_{[-1,1]}, \qquad x \in (-1,1), \tag{1.1}$$

and the Markov inequality [14]

$$\|P_n'\|_{[-1,1]} \le n^2 \|P_n\|_{[-1,1]} \tag{1.2}$$

for estimating the derivative of polynomials P_n of degree at most n in terms of the supremum norm $\|P_n\|_{[-1,1]}$ of the polynomials. In (1.1) the order of the right hand side is n, and the estimate can be used at inner points of $[-1,1]$. In (1.2) the growth of the right-hand side is n^2, which is much larger, but (1.2) can also be used close to the endpoints ± 1, and it gives a global estimate. We shall use the terminology "Bernstein-type inequality" for estimating the derivative away from endpoints with a factor of order n, and "Markov-type inequality" for a global estimate on the derivative with a factor of order n^2.

The Bernstein and Markov inequalities have been generalized and improved in several directions over the last century, see the extensive books [3] and [15]. See also [6] and the references there for various improvements. For rational functions sharp Bernstein-type inequalities have been given for circles [4] and for compact subsets of the real line and circles, see [4], [7], [13]. We are unaware of a corresponding Markov-type estimate. General (but not sharp) estimates on the derivative of rational functions can also be found in [20] and [21].

The aim of this paper is to give the sharp form of the Bernstein and Markov inequalities for rational functions on smooth Jordan curves and arcs. We shall be primarily interested in the asymptotically best possible estimates and in the structure of the constants on the right hand side. As we shall see, from this point of view there is a huge difference between Jordan curves and Jordan arcs. All the results are formulated in terms of the normal derivatives of certain Green's functions with poles at the poles of the rational functions in question. When all the poles are at infinity we recapture the corresponding results for polynomials that have been proven in the last decade.

We shall use basic notions of potential theory, for the necessary background we refer to the books [1], [18], [22] or [25].

2 Results

We shall work with Jordan curves and Jordan arcs on the plane. Recall that a Jordan curve is a homeomorphic image of a circle, while a Jordan arc is a homeomorphic image of a segment. We say that the Jordan arc Γ is C^2 smooth if it has a parametrization $\gamma(t)$, $t \in [-1,1]$, which is twice continuously differentiable and $\gamma'(t) \neq 0$ for $t \in [-1,1]$. Similarly we speak of C^2 smoothness of a Jordan curve, the only difference is that for a Jordan curve the parameter domain is the unit circle.

If Γ is a Jordan curve, then we think it counterclockwise oriented. $\overline{\mathbf{C}} \setminus \Gamma$ has two connected components, we denote the bounded component by G_- and the unbounded one by G_+. At a point $z \in \Gamma$ we denote the two normals to Γ by $\mathbf{n}_\pm = \mathbf{n}_\pm(z)$ with the agreement that $\mathbf{n}_-$ points towards G_-. So, as we move on Γ according to its orientation, $\mathbf{n}_-$ is the left and $\mathbf{n}_+$ is the right normal. In a similar fashion, if Γ is a Jordan arc then we take an orientation of Γ and let $\mathbf{n}_-$ resp. $\mathbf{n}_+$ denote the left resp. right normal to Γ with respect to this orientation.

Let R be a rational function. We say it has total degree n if the sum of the order of its poles (including the possible pole at ∞) is n. We shall often use summations $\sum_a$ where a runs through the poles of R, and let us agree that in such sums a pole a appears as many times as its order.

In this paper we determine the asymptotically sharp analogues of the Bernstein and Markov inequalities on Jordan curves and arcs Γ for rational functions. Note however, that even in the simplest case $\Gamma = [-1,1]$ there is no Bernstein- or Markov-type inequality just in terms of the degree of the rational function. Indeed, if $M > 0$, then $R_2(z) = 1/(1 + Mz^2)$ is at most 1 in absolute value on $[-1,1]$, but $|R_2'(1/\sqrt{M})| = \sqrt{M}/2$, which can be arbitrary large if M is large. Therefore, to get Bernstein-Markov-type inequalities in the classical sense we should limit the poles of R to lie far from Γ. In this paper we assume that the poles of the rational functions lie in a closed set $Z \subset \overline{\mathbf{C}} \setminus \Gamma$ which we fix in advance. If $Z = \{\infty\}$, then R has to be a polynomial.

In what follows $\|f\|_\Gamma = \sup_{z\in\Gamma} |f(z)|$ denotes the supremum norm on Γ, and $g_G(z,a)$ the Green's function of a domain G with pole at $a \in G$.

Our first result is a Bernstein-type inequality on Jordan curves.

Theorem 2.1 *Let Γ be a C^2 smooth Jordan curve on the plane, and let R_n be a rational function of total degree n such that its poles lie in the fixed closed set $Z \subset \overline{\mathbf{C}} \setminus \Gamma$. If $z_0 \in \Gamma$, then*

$$|R_n'(z_0)| \leq (1 + o(1))\|R_n\|_\Gamma \max\left(\sum_{a\in Z\cap G_+} \frac{\partial g_{G_+}(z_0,a)}{\partial \mathbf{n}_+}, \sum_{a\in Z\cap G_-} \frac{\partial g_{G_-}(z_0,a)}{\partial \mathbf{n}_-} \right), \tag{2.1}$$

where the summation is for the poles of R_n and where $o(1)$ denotes a quantity that tends to 0 uniformly in R_n as $n \to \infty$. Furthermore, this estimate holds uniformly in $z_0 \in \Gamma$.

The normal derivative $\partial g_{G_\pm}(z_0,a)/\partial \mathbf{n}_\pm$ is 2π-times the density of the harmonic measure of a in the domain $G_\pm$, where the density is taken with respect to the arc measure on Γ. Thus, the right hand side in (2.1) is easy to formulate in terms of harmonic measures, as well.

Corollary 2.2 *If Γ is as in Theorem 2.1 and P_n is a polynomial of degree at most n, then for $z_0 \in \Gamma$ we have*

$$|P_n'(z_0)| \le (1+o(1))n\|P_n\|_\Gamma \frac{\partial g_{G_+}(z_0,\infty)}{\partial \mathbf{n}_+}. \tag{2.2}$$

This is Theorem 1.3 in the paper [16]. The estimate (2.2) is asymptotically the best possible (see below), and on the right $\partial g_{G_+}(z_0,\infty)/\partial \mathbf{n}_+$ is 2π-times of the density of the equilibrium measure of Γ with respect to the arc measure on Γ. Therefore, the corollary shows an explicit relation in between the Bernstein factor at a given point and the harmonic density at the same point.

If R_n has order $n + o(n)$ and we take the sum on the right of (2.1) only on some of its n poles, then (2.1) still holds (i.e. $o(n)$ poles do not have to be accounted for). Now in this sense Theorem 2.1 is sharp.

Theorem 2.3 *Let Γ be as in Theorem 2.1 and let $Z \subset \overline{\mathbf{C}} \setminus \Gamma$ be a non-empty closed set. If $\{a_{1,n}, \dots, a_{n,n}\}$, $n = 1, 2, \dots$ is an array of points from Z and $z_0 \in \Gamma$ is a point on Γ, then there are non-zero rational functions R_n of degree $n + o(n)$ such that $a_{1,n}, \dots, a_{n,n}$ are among the poles of R_n and*

$$|R_n'(z_0)| \ge (1-o(1))\|R_n\|_\Gamma \max\left(\sum_{a_{j,n}\in G_+} \frac{\partial g_{G_+}(z_0,a_{j,n})}{\partial \mathbf{n}_+}, \sum_{a_{j,n}\in G_-} \frac{\partial g_{G_-}(z_0,a_{j,n})}{\partial \mathbf{n}_-}\right). \tag{2.3}$$

In this theorem if a point $a \in Z$ appears k times in $\{a_{1,n}, \dots, a_{n,n}\}$, then the understanding is that at a the rational function R_n has a pole of order k.

Next, we consider the Bernstein-type inequality for rational functions on a Jordan arc.

Theorem 2.4 *Let Γ be a C^2 smooth Jordan arc on the plane, and let R_n be a rational function of total degree n such that its poles lie in the fixed closed set $Z \subset \overline{\mathbf{C}} \setminus \Gamma$. If $z_0 \in \Gamma$ is different from the endpoints of Γ, then*

$$|R_n'(z_0)| \le (1+o(1))\|R_n\|_\Gamma \max\left(\sum_{a\in Z} \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z_0,a)}{\partial \mathbf{n}_+}, \sum_{a\in Z} \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z_0,a)}{\partial \mathbf{n}_-}\right), \tag{2.4}$$

where the summation is for the poles of R_n and where $o(1)$ denotes a quantity that tends to 0 uniformly in R_n as $n \to \infty$. Furthermore, (2.4) holds uniformly in $z_0 \in J$ for any closed subarc J of Γ that does not contain either of the endpoints of Γ.

Corollary 2.5 *If Γ is as in Theorem 2.4 and P_n is a polynomial of degree at most n, then for $z_0 \in \Gamma$, which is different from the endpoints of Γ, we have*

$$|P_n'(z_0)| \le (1+o(1))n\|P_n\|_\Gamma \max\left(\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z_0,\infty)}{\partial \mathbf{n}_+}, \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z_0,\infty)}{\partial \mathbf{n}_-}\right). \tag{2.5}$$

This was proven in [11] for analytic Γ and in [24] for C^2 smooth Γ. More generally, if $a_1, \ldots, a_m$ are finitely many fixed points outside Γ and

$$R_n(z) = P_{n_0,0}(z) + \sum_{i=1}^m P_{n_i,i}\left(\frac{1}{z-a_i}\right) \tag{2.6}$$

where $P_{n_i,i}$ are polynomials of degree at most n_i, then, as $n_0 + \cdots n_m \to \infty$,

$$|R_n'(z_0)| \le (1+o(1))\|R_n\|_\Gamma \max\left(\sum_{i=0}^m n_i \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z_0,a_i)}{\partial \mathbf{n}_+}, \sum_{i=0}^m n_i \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z_0,a_i)}{\partial \mathbf{n}_-}\right), \tag{2.7}$$

where $a_0 = \infty$.

Theorem 2.4 is sharp again regarding the Bernstein factor on the right.

Theorem 2.6 *Let Γ be as in Theorem 2.4 and let $Z \subset \overline{\mathbf{C}} \setminus \Gamma$ be a non-empty closed set. If $\{a_{1,n}, \ldots, a_{n,n}\}$, $n = 1, 2, \ldots$ is an arbitrary array of points from Z and $z_0 \in \Gamma$ is any point on Γ different from the endpoints of Γ, then there are non-zero rational functions R_n of degree $n + o(n)$ such that $a_{1,n}, \ldots, a_{n,n}$ are among the poles of R_n and*

$$|R_n'(z_0)| \ge (1-o(1))\|R_n\|_\Gamma \max\left(\sum_{a\in Z} \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z_0,a)}{\partial \mathbf{n}_+}, \sum_{a\in Z} \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z_0,a)}{\partial \mathbf{n}_-}\right). \tag{2.8}$$

Now we consider the Markov-type inequality on a C^2 Jordan arc Γ for rational functions of the form (2.6). Let A, B be the two endpoints of Γ. We need the quantity

$$\Omega_a(A) = \lim_{z\to A,\ z\in\Gamma} \sqrt{|z-A|}\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z,a)}{\partial \mathbf{n}_\pm(z)}. \tag{2.9}$$

It will turn out that this limit exists and it is the same if we use in it the left or the right normal derivative (i.e. it is indifferent if we use $\mathbf{n}_+$ or $\mathbf{n}_-$ in the definition). We define $\Omega_a(B)$ similarly. With these we have

Theorem 2.7 *Let Γ be a C^2 smooth Jordan arc on the plane, and let R_n be a rational function of total degree n of the form (2.6) with fixed $a_0, a_1, \ldots, a_m$. Then*

$$\|R_n'\|_\Gamma \le (1+o(1))\|R_n\|_\Gamma 2\max\left(\sum_{i=0}^m n_i\Omega_{a_i}(A), \sum_{i=0}^m n_i\Omega_{a_i}(B)\right)^2, \tag{2.10}$$

where $o(1)$ tends to 0 uniformly in R_n as $n \to \infty$.

Theorem 2.7 is again the best possible, but we shall not state that, for we will have a more general result in Theorem 2.8.

Actually, there is a separate Markov-type inequality around both endpoints A and B. Indeed, let U be a closed neighborhood of A that does not contain B. Then

$$\|R_n'\|_{\Gamma\cap U} \le (1+o(1))\|R_n\|_\Gamma 2\left(\sum_{i=0}^m n_i\Omega_{a_i}(A)\right)^2, \tag{2.11}$$

and this is sharp. Now (2.10) is clearly a consequence of this and its analogue for the endpoint B. Note that the discussion below will show that the right-hand side in (2.10) is of size $\sim n^2$, while on any closed Jordan subarc of Γ that does not contain A or B the derivative R_n' is $O(n)$.

Let us also mention that in these theorems in general the $o(1)$ term in the $1+o(1)$ factors on the right cannot be omitted. Indeed, consider for example, Corollary 2.2. It is easy to construct a C^2 Jordan curve for which the normal derivative on the right of (2.2) is small, so with $P_1(z)=z$ the inequality in (2.2) fails if we write 0 instead of $o(1)$.

It is also interesting to consider higher derivatives, though we can do a complete analysis only for rational functions of the form (2.6). For them the inequalities (2.1) and (2.4) can simply be iterated. For example, if Γ is a Jordan arc, then under the assumptions of Theorem 2.4 we have for any fixed $k = 1,2,\dots$

$$|R_n^{(k)}(z_0)| \le (1+o(1))\|R_n\|_\Gamma \max\left(\sum_{i=0}^m n_i\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z_0,a_i)}{\partial \mathbf{n}_+}, \sum_{i=0}^m n_i\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z_0,a_i)}{\partial \mathbf{n}_-}\right)^k \tag{2.12}$$

uniformly in $z_0\in J$ where J is any closed subarc of Γ that does not contain the endpoints of Γ. It can also be proven that this inequality is sharp for every k and every $z_0\in\Gamma$ in the sense given in Theorems 2.3 and 2.6.

The situation is different for the Markov inequality (2.10), because if we iterate it, then we do not obtain the sharp inequality for the norm of the k-th derivative (just like the iteration of the classical A. A. Markov inequality does not give the sharp V. A. Markov inequality for higher derivatives of polynomials). Indeed, the sharp form is given in the following theorem.

Theorem 2.8 *Let Γ be a C^2 smooth Jordan arc on the plane, and let R_n be a rational function of total degree n of the form (2.6) with fixed $a_0, a_1, \dots, a_m$. Then for any fixed $k = 1,2,\dots$ we have*

$$\|R_n^{(k)}\|_\Gamma \le (1+o(1))\|R_n\|_\Gamma \frac{2^k}{(2k-1)!!}\max\left(\sum_{i=0}^m n_i\Omega_{a_i}(A), \sum_{i=0}^m n_i\Omega_{a_i}(B)\right)^{2k}, \tag{2.13}$$

where $o(1)$ tends to 0 uniformly in R_n as $n\to\infty$. Furthermore, this is sharp, for one cannot write a constant smaller than 1 instead of $1+o(1)$ on the right.

Recall that $(2k-1)!! = 1 \cdot 3 \cdot \dots \cdot (2k-3) \cdot (2k-1)$.

As before, this theorem will follow if we prove for any closed neighborhood U of the endpoint A that does not contain the other endpoint B the estimate

$$\|R_n^{(k)}\|_{\Gamma \cap U} \leq (1+o(1))\|R_n\|_\Gamma \frac{2^k}{(2k-1)!!} \left(\sum_{i=0}^m n_i \Omega_{a_i}(A)\right)^{2k}. \tag{2.14}$$

Corollary 2.9 *If Γ is as in Theorem 2.8 and P_n is a polynomial of degree at most n, then*

$$\|P_n^{(k)}\|_\Gamma \leq (1+o(1))\|P_n\|_\Gamma \frac{2^k}{(2k-1)!!} n^{2k} \max\left(\Omega_\infty(A), \Omega_\infty(B)\right)^{2k}. \tag{2.15}$$

This was proven in [24, Theorem 2].

The outline of the paper is as follows.

- After some preparations first we verify Theorem 2.1 (Bernstein-type inequality) for *analytic curves* via conformal maps onto the unit disk and using on the unit disk a result of Borwein and Erdélyi. This part uses in an essential way a decomposition theorem for meromorphic functions.
- Next, Theorem 2.4 is verified for *analytic arcs* from the analytic case of Theorem 2.1 for Jordan curves via the Joukowskii mapping.
- For C^2 arcs Theorem 2.4 follows from its version for analytic arcs by an appropriate approximation.
- For C^2 curves Theorem 2.1 will be deduced from Theorem 2.4 by introducing a gap (omitting a small part) on the given Jordan curve to get a Jordan arc, and then by closing up that gap.
- The Markov-type inequality Theorem 2.8 is deduced from the Bernstein-type inequality on arcs (Theorem 2.4, more precisely from its higher derivative variant (2.12)) by a symmetrization technique during which the given endpoint where we consider the Markov-type inequality is mapped into an inner point of a different Jordan arc.
- Finally, in Section 10 we prove the sharpness of the theorems using conformal maps and sharp forms of Hilbert's lemniscate theorem.

3 Preliminaries

In this section we collect some tools that are used at various places in the proofs.

3.1 A "rough" Bernstein-type inequality

We need the following "rough" Bernstein-type inequality on Jordan curves.

Proposition 3.1 *Let Γ be a C^2 smooth Jordan curve and $Z \subset \overline{\mathbf{C}} \setminus \Gamma$ a closed set. Then there exists $C > 0$ such that for any rational function R_n with poles in Z and of degree n, we have*

$$\|R_n'\|_\Gamma \leq Cn \|R_n\|_\Gamma .$$

Proof. Recall that G_- denotes the inner, while G_+ denotes the outer domain to Γ. We shall need the following Bernstein-Walsh-type estimate:

$$|R_n(z)| \leq \|R_n\|_\Gamma \exp\left(\sum_{a \in Z \cap G_\pm} g_{G_\pm}(z,a)\right) \tag{3.1}$$

where the summation is taken for $a \in Z \cap G_+$ if $z \in G_+$ (and then g_{G_+} is used) and for $a \in Z \cap G_-$ if $z \in G_-$. Indeed, suppose, for example, that $z \in G_-$. The function

$$\log|R_n(z)| - \left(\sum_{a \in Z \cap G_-} g_{G_-}(z,a)\right)$$

is subharmonic in G_- and has boundary values $\leq \log\|R_n\|_\Gamma$ on Γ, so (3.1) follows from the maximum principle for subharmonic functions.

Let $z_0 \in \Gamma$ be arbitrary. It follows from Proposition 3.10 below that there is a $\delta > 0$ such that for $\mathrm{dist}(z,\Gamma) < \delta$ we have for all $a \in Z$ the bound

$$g_{G_\pm}(z,a) \leq C_1 \mathrm{dist}(z,\Gamma) \leq C_1 |z - z_0|$$

with some constant C_1.

Let $C_{1/n}(z_0) := \{z \mid |z - z_0| = 1/n\}$ be the circle about z_0 of radius $1/n$ (assuming $n > 2/\delta$). For $z \in C_{1/n}(z_0)$ the sum on the right of (3.1) can be bounded as

$$\sum_{a \in Z \cap G_+} g_{G_+}(z,a) \leq nC_1 |z - z_0| \leq C_1$$

if $z \in G_+$, and a similar estimate holds if $z \in G_-$. Therefore, $|R_n(z)| \leq e^{C_1}\|R_n\|_\Gamma$.

Now we apply Cauchy's integral formula

$$|R_n'(z_0)| = \left|\frac{1}{2\pi i}\int_{C_{1/n}(z_0)} \frac{R_n(z)}{(z-z_0)^2}dz\right| \leq \frac{1}{2\pi}\frac{2\pi}{n}\frac{\|R_n\|_\Gamma e^{C_1}}{n^{-2}} = \|R_n\|_\Gamma n e^{C_1},$$

which proves the proposition.

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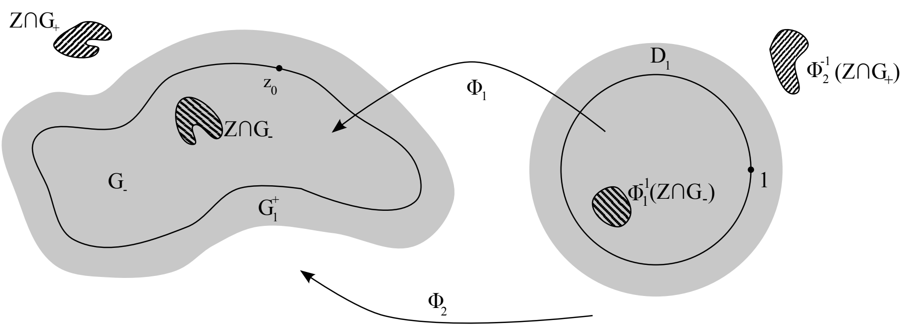

Figure 1: The two conformal mappings Φ_1, Φ_2, the domain D_1 and the possible location of poles

3.2 Conformal mappings onto the inner and outer domains

Denote $\mathbf{D} = \{v \mid \ |v| < 1\}$ the unit disk and $\mathbf{D}_+ = \{v \mid \ |v| > 1\} \cup \{\infty\}$ its exterior.

By the Kellogg-Warschawski theorem (see e.g. [17, Theorem 3.6]), if Γ is C^2 smooth, then Riemann mappings from $\mathbf{D}, \mathbf{D}_+$ onto G_-, G_+, respectively, as well as their derivatives can be extended continuously to the boundary Γ. Under analyticity assumption, the corresponding Riemann mappings have extensions to larger domains. In fact, the following proposition holds (see e.g. Proposition 7 in [11] with slightly different notation).

Proposition 3.2 *Assume that Γ is analytic, and let $z_0 \in \Gamma$ be fixed. Then there exist two Riemann mappings $\Phi_1 : \mathbf{D} \to G_-$, $\Phi_2 : \mathbf{D}_+ \to G_+$ such that $\Phi_j(1) = z_0$ and $\left|\Phi_j'(1)\right| = 1$, $j = 1, 2$. Furthermore, there exist $0 \leq r_2 < 1 < r_1 \leq \infty$ such that Φ_1 extends to a conformal map of $D_1 := \{v \mid \ |v| < r_1\}$ and Φ_2 extends to a conformal map of $D_2 := \{v \mid \ |v| > r_2\} \cup \{\infty\}$.*

Since the argument of $\Phi_j'(1)$ gives the angle of the tangent line to Γ at z_0, the arguments of $\Phi_1'(1)$ and of $\Phi_2'(1)$ must be the same, which combined with $|\Phi_1'(1)| = |\Phi_2'(1)| = 1$ yields $\Phi_1'(1) = \Phi_2'(1)$. Therefore,

$$\Phi_1(1) = \Phi_2(1) = z_0, \ \Phi_1'(1) = \Phi_2'(1), \ |\Phi_1'(1)| = |\Phi_2'(1)| = 1. \tag{3.2}$$

From now on, for a given $z_0 \in \Gamma$ we fix these two conformal maps. These mappings and the corresponding domains are depicted on Figure 1. We may assume that D_1 and $\Phi_2^{-1}(Z) \cap G_+$, as well as D_2 and $\Phi_1^{-1}(Z) \cap G_-$ are of positive distance from one another (by slightly decreasing r_1 and increasing r_2, if necessary).

Proposition 3.3 *The following hold for arbitrary $a \in G_-$, $b \in G_+$ with $a' := \Phi_1^{-1}(a)$, $b' := \Phi_2^{-1}(b)$*

$$\frac{\partial g_{G_-}(z_0, a)}{\partial \mathbf{n}_-} = \frac{\partial g_{\mathbf{D}}(1, a')}{\partial \mathbf{n}_-} = \frac{1 - |a'|^2}{|1 - a'|^2},$$

$$\frac{\partial g_{G_+}(z_0, b)}{\partial \mathbf{n}_+} = \frac{\partial g_{\mathbf{D}_+}(1, b')}{\partial \mathbf{n}_+} = \frac{|b'|^2 - 1}{|1 - b'|^2}, \qquad \text{if } b' \neq \infty,$$

and if $b' = \infty$, then

$$\frac{\partial g_{G_+}(z_0, b)}{\partial \mathbf{n}_+} = \frac{\partial g_{\mathbf{D}_+}(1, \infty)}{\partial \mathbf{n}_+} = 1.$$

This proposition is a slight generalization of Proposition 8 from [11] with the same proof.

3.3 The Borwein-Erdélyi inequality

The following inequality will be central in establishing Theorem 2.1 in the analytic case, it serves as a model. For a proof we refer to [4] (see also [3, Theorem 7.1.7]).

Let $\mathbf{T}$ denote the unit circle.

Proposition 3.4 (Borwein-Erdélyi) *Let $a_1, \ldots, a_m \in \mathbf{C} \setminus \mathbf{T}$ and let*

$$B_m^+(v) := \sum_{|a_j|>1} \frac{|a_j|^2 - 1}{|a_j - v|^2}, \qquad B_m^-(v) := \sum_{|a_j|<1} \frac{1 - |a_j|^2}{|a_j - v|^2},$$

and $B_m(v) := \max(B_m^+(v), B_m^-(v))$. If P is a polynomial with $\deg(P) \leq m$ and $R_m(v) = P(v) / \prod_{j=1}^m (v - a_j)$ is a rational function, then

$$|R_m'(v)| \leq B_m(v) \, ||R_m||_{\mathbf{T}}, \qquad v \in \mathbf{T}.$$

Since the Green's function $g_{\mathbf{D}}(z, a)$, $a \in \mathbf{D}$, is $\log(|1 - \overline{a}z|/|z - a|)$, simple computation shows that

$$\frac{1 - |a|^2}{|a - v|^2} = \frac{\partial g_{\mathbf{D}}(v, a)}{\partial \mathbf{n}_-},$$

and a similar relation is true for the outer domain $\mathbf{D}_+$ and for $|a| > 1$. Hence, Proposition 3.4 can be written as follows (see [11, Theorem 4]).

Proposition 3.5 *Let $R_m(v) = P(v)/Q(v)$ be an arbitrary rational function with no poles on the unit circle, where P and Q are polynomials. Denote the poles of R_m by $a_1, \ldots, a_m$, where each pole is repeated as many times as its order. Then, for $v \in \mathbf{T}$,*

$$|R_m'(v)| \leq ||R_m||_{\mathbf{T}} \cdot \max\left(\sum_{|a_j|>1} \frac{\partial g_{\mathbf{D}_+}(v, a_j)}{\partial \mathbf{n}_+}, \sum_{|a_j|<1} \frac{\partial g_{\mathbf{D}}(v, a_j)}{\partial \mathbf{n}_-} \right).$$

3.4 A Gonchar-Grigorjan type estimate

It is a standard fact that a meromorphic function on a domain with finitely many poles can be decomposed into the sum of an analytic function and a rational function (which is the sum of the principal parts at the poles). If the rational function is required to vanish at ∞, then this decomposition is unique.

L.D. Grigorjan with A.A. Gonchar investigated in a series of papers the supremum norm of the sum of the principal parts of a meromorphic function on the boundary of the given domain in terms of the supremum norm of the function itself. In particular, Grigorjan showed in [9] that if $K \subset \mathbf{D}$ is a fixed compact subset of the unit disk $\mathbf{D}$, then there exists a constant $C > 0$ such that all meromorphic functions f on $\mathbf{D}$ having poles only in K have principal part R (with $R(\infty) = 0$) for which $\|R\| \le C \log n \|f\|$, where n is the sum of the order of the poles of f (here $\|f\| := \limsup_{|\zeta| \to 1-} |f(\zeta)|$).

The following recent result (which is [10, Theorem 1]) generalizes this to more general domains.

Proposition 3.6 *Suppose that $D \subset \overline{\mathbf{C}}$ is a finitely connected domain such that its boundary ∂D consists of finitely many disjoint C^2 smooth Jordan curves. Let $Z \subset D$ be a closed set, and suppose that $f : D \to \overline{\mathbf{C}}$ is a meromorphic function on D such that all of its poles are in Z. Denote the total order of the poles of f by n. If f_r is the sum of the principal parts of f (with $f_r(\infty) = 0$) and f_a is its analytic part (so that $f = f_r + f_a$), then*

$$\|f_r\|_{\partial D}, \|f_a\|_{\partial D} \le C \log n \, \|f\|_{\partial D},$$

where the constant $C = C(D, Z) > 0$ depends only on D and Z.

In this statement

$$\|f\|_{\partial D} := \limsup_{\zeta \in D, \ \zeta \to \partial D} |f(\zeta)|,$$

but we shall apply the proposition in cases when f is actually continuous on ∂D.

3.5 A Bernstein-Walsh-type approximation theorem

We shall use the following approximation theorem.

Proposition 3.7 *Let τ be a Jordan curve and K a compact subset of its interior domain. Then there are a $C > 0$ and $0 < q < 1$ with the following property. If f is analytic inside τ such that $|f(z)| \le M$ for all z, then for every $w_0 \in K$ and $m = 1, 2, \ldots$ there are polynomials S_m of degree at most m such that $S_m(w_0) = f(w_0)$, $S'_m(w_0) = f'(w_0)$ and*

$$\|f - S_m\|_K \le CMq^m. \tag{3.3}$$

Proof. Let τ_1 be a lemniscate, i.e. the level curve of a polynomial, say $\tau_1 = \{z \,|\, |T_N(z)| = 1\}$, such that τ_1 lies inside τ and K lies inside τ_1. According to Hilbert's lemniscate theorem (see e.g. [18, Theorem 5.5.8]) there is such a τ_1. Then K is contained in the interior domain of $\tau_\theta = \{z \mid |T_N(z)| = \theta\}$ for some $\theta < 1$. By Theorem 3 in [26, Sec. 3.3] (or use [18, Theorem 6.3.1]) there are polynomials R_m of degree at most $m = 1, 2, \ldots$ such that

$$\|f - R_m\|_{\tau_\theta} \leq C_1 M q^m \tag{3.4}$$

with some C_1 and $q < 1$ (the q depends only on θ and the degree N of T_N). Actually, in that theorem the right hand side does not show M explicitly, but the proof, in particular the error formula (12) in [26, Section 3.3] (or the error formula (6.9) in [18, Section 6.3]), gives (3.4).

Now (3.4) pertains to hold also on the interior domain to τ_θ, so if δ is the distance in between K and τ_θ and $w_0 \in K$, then for all $|\xi - w_0| = \delta$ we have $|f(\xi) - R_m(\xi)| \leq C_1 M q^m$. Hence, by Cauchy's integral formula for the derivative we have

$$|f'(w_0) - R_m'(w_0)| \leq \frac{C_1 M q^m}{\delta}.$$

Therefore, the polynomial

$$S_m(z) = R_m(z) + (f(w_0) - R_m(w_0)) + (f'(w_0) - R_m'(w_0))(z - w_0)$$

satisfies the requirements with $C = C_1(2 + \mathrm{diam}(K)/\delta)$ in (3.3).

■

3.6 Bounds and smoothness for Green's functions

In this section we collect some simple facts on Green's functions and their normal derivatives.

Let $K \subset \overline{\mathbf{C}}$ be a compact set with connected complement and $Z \subset \overline{\mathbf{C}} \setminus K$ a closed set. Suppose that σ is a Jordan curve that separates K and Z, say K lies in the interior of σ while Z lies in its exterior. Assume also that there is a family $\{\gamma_\tau\} \subset K$ of Jordan arcs such that $\mathrm{diam}(\gamma_\tau) \geq d > 0$ with some $d > 0$, where $\mathrm{diam}(\gamma_\tau)$ denotes the diameter of γ_τ.

First we prove

Proposition 3.8 *There are $c_0, C_0 > 0$ such that for all τ, $z \in \sigma$ and all $a \in Z$ we have*

$$c_0 \leq g_{\overline{\mathbf{C}} \setminus \gamma_\tau}(z, a) \leq C_0. \tag{3.5}$$

Proof. We have the formula ([18, p. 107])

$$g_{\overline{\mathbf{C}} \setminus \gamma_\tau}(z, \infty) = \log \frac{1}{\mathrm{cap}(\gamma_\tau)} + \int \log |z - t| d\mu_{\gamma_\tau}(t),$$

where μ_{γ_τ} is the equilibrium measure of γ_τ and where $\mathrm{cap}(\gamma_\tau)$ denotes the logarithmic capacity of γ_τ. Since (see [18, Theorem 5.3.2])

$$\mathrm{cap}(\gamma_\tau) \geq \frac{\mathrm{diam}(\gamma_\tau)}{4} \geq \frac{d}{8},$$

and for $z \in \sigma$, $t \in \gamma_\tau$ we have $|z - t| \leq \mathrm{diam}(\sigma)$, we obtain

$$g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(z, \infty) \leq \log\frac{1}{d/8} + \log\mathrm{diam}(\sigma) =: C_1.$$

Let Ω be the exterior of σ (including ∞). By Harnack's inequality ([18, Corollary 1.3.3]) for any closed set $Z \subset \Omega$ there is a constant C_Z such that for all positive harmonic functions u on Ω we have

$$\frac{1}{C_Z}u(\infty) \leq u(a) \leq C_Z u(\infty), \qquad a \in Z.$$

Apply this to the harmonic function $g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(z, a) = g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(a, z)$ (recall that Green's functions are symmetric in their arguments), $z \in \sigma$, $a \in Z$, to conclude for $z \in \sigma$

$$g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(z, a) = g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(a, z) \leq C_Z g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(\infty, z) = C_Z g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(z, \infty) \leq C_Z C_1.$$

To prove a lower bound note that

$$g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(z, \infty) \geq g_{\overline{\mathbf{C}}\setminus K}(z, \infty) \geq c_1, \qquad z \in \sigma,$$

because $\gamma_\tau \subset K$ and $g_{\overline{\mathbf{C}}\setminus K}(z, \infty)$ is a positive harmonic function outside K. From here we get

$$g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(z, a) \geq \frac{c_1}{C_Z}, \qquad z \in \sigma,\ a \in Z,$$

exactly as before by appealing to the symmetry of the Green's function and to Harnack's inequality.

∎

Corollary 3.9 *With the c_0, C_0 from the preceding lemma for all τ, $a \in Z$ and for all z lying inside σ we have*

$$\frac{c_0}{C_0} g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(z, \infty) \leq g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(z, a) \leq \frac{C_0}{c_0} g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(z, \infty). \tag{3.6}$$

Proof. For $z \in \sigma$ the inequality (3.6) was shown in the preceding proof. Since both $g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(z, \infty)$ and $g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(z, a)$ are harmonic in the domain that lies in between γ_τ and σ and both vanish on γ_τ, the statement follows from the maximum principle.

∎

Next, let Γ be a C^2 Jordan curve and $G_\pm$ the interior and exterior domains to Γ (see Section 2). Assume, as before, that $Z \subset \overline{\mathbf{C}} \setminus \Gamma$ is a closed set.

Proposition 3.10 *There are constants $C_1, c_1 > 0$ such that*

$$c_1 \leq \frac{\partial g_{G_-}(z_0, a)}{\partial \mathbf{n}_-} \leq C_1, \qquad a \in Z \cap G_- \tag{3.7}$$

and

$$c_1 \leq \frac{\partial g_{G_+}(z_0, a)}{\partial \mathbf{n}_+} \leq C_1, \qquad a \in Z \cap G_+. \tag{3.8}$$

These bounds hold uniformly in $z_0 \in \Gamma$. Furthermore, the Green's functions $g_{G_\pm}(z, a)$, $a \in Z$, are uniformly Hölder 1 equi-continuous close to the boundary Γ.

Proof. It is enough to prove (3.7). Let $b_0 \in G_-$ be a fixed point and let φ be a conformal map from the unit disk $\mathbf{D}$ onto G_- such that $\varphi(0) = b_0$. By the Kellogg-Warschawski theorem (see [17, Theorem 3.6]) φ' has a continuous extension to the closed unit disk which does not vanish there. It is clear that $g_{G_-}(z, b_0) = -\log|\varphi^{-1}(z)|$, and consider some local branch of $-\log\varphi^{-1}(z)$ for z lying close to z_0. By the Cauchy-Riemann equations

$$\frac{\partial g_{G_-}(z_0, b_0)}{\partial \mathbf{n}_-} = \left| \left(-\log\varphi^{-1}(z)\right)' \Big|_{z = z_0} \right|$$

(note that the directional derivative of g_{G_-} in the direction perpendicular to $\mathbf{n}_-$ has 0 limit at $z_0 \in \partial G_-$), so we get the formula

$$\frac{\partial g_{G_-}(z_0, b_0)}{\partial \mathbf{n}_-} = \frac{1}{|\varphi'(\varphi^{-1}(z_0))|}, \tag{3.9}$$

which shows that this normal derivative is finite, continuous in $z_0 \in \Gamma$ and positive.

Let now σ be a Jordan curve that separates $(Z \cap G_-) \cup \{b_0\}$ from Γ. Map G_- conformally onto $\overline{\mathbf{C}} \setminus [-1, 1]$ by a conformal map Φ so that $\Phi(b_0) = \infty$. Then $g_{G_-}(z, a) = g_{\overline{\mathbf{C}} \setminus [-1,1]}(\Phi(z), \Phi(a))$, and $\Phi(\sigma)$ is a Jordan curve that separates $\Phi((Z \cap G_-) \cup \{b_0\})$ from $[-1, 1]$. Now apply Proposition 3.8 to $\overline{\mathbf{C}} \setminus [-1, 1]$ and to $\Phi(\sigma)$ to conclude that all the Green's functions $g_{\overline{\mathbf{C}} \setminus [-1,1]}(w, \Phi(a))$, $a \in Z \cup \{b_0\}$, are comparable on $\Phi(\sigma)$ in the sense that all of them lie in between two positive constants $c_2 < C_2$ there. In view of what we have just said, this means that the Green's functions $g_{G_-}(z, a)$, $a \in Z \cup \{b_0\}$, are comparable on σ in the sense that all of them lie in between the same $c_2 < C_2$ there. But then they are also comparable in the domain that lies in between Γ and σ, and hence

$$\frac{c_2}{C_2} \frac{\partial g_{G_-}(z_0, b_0)}{\partial \mathbf{n}_-} \leq \frac{\partial g_{G_-}(z_0, a)}{\partial \mathbf{n}_-} \leq \frac{C_2}{c_2} \frac{\partial g_{G_-}(z_0, b_0)}{\partial \mathbf{n}_-}, \qquad a \in Z,$$

which proves (3.7) in view of (3.9).

The uniform Hölder continuity is also easy to deduce from (3.9) if we compose φ by fractional linear mappings of the unit disk onto itself (to move the pole $\varphi(0)$ to other points).

■

4 The Bernstein-type inequality on analytic curves

In this section we assume that Γ is analytic, and prove (2.1) using Propositions 3.5, 3.6 and 3.7.

Fix $z_0 \in \Gamma$ and consider the conformal maps Φ_1 and Φ_2 from Section 3.2. Recall that the inner map Φ_1 has an extension to a disk $D_1 = \{z \mid |z| < r_1\}$ and the external map Φ_2 has an extension to the exterior $D_2 = \{z \mid |z| > r_2\}$ of a disk with some $r_2 < 1 < r_1$. For simpler notation, in what follows we shall assume that Φ_1 resp. Φ_2 actually have extensions to a neighborhood of the closures $\overline{D_1}$ resp. $\overline{D_2}$ (which can be achieved by decreasing r_1 and increasing r_2 if necessary).

In what follows we set $\mathbf{T}(r) = \{z \mid |z| = r\}$ for the circle of radius r about the origin. As before, $\mathbf{T} = \mathbf{T}(1)$ denotes the unit circle.

The constants C, c below depend only on Γ and they are not the same at each occurrence.

We decompose R_n as,

$$R_n = f_1 + f_2$$

where f_1 is a rational function with poles in $Z \cap G_-$, $f_1(\infty) = 0$ and f_2 is a rational function with poles in $Z \cap G_+$. This decomposition is unique. If we put $N_1 := \deg(f_1)$, $N_2 := \deg(f_2)$, then $N_1 + N_2 = n$. Denote the poles of f_1 by α_j, $j = 1, \ldots, N_1$, and the poles of f_2 by β_j, $j = 1, \ldots, N_2$ (with counting the orders of the poles).

We use Proposition 3.6 on G_- to conclude

$$\|f_1\|_\Gamma, \|f_2\|_\Gamma \le C \log n \|R_n\|_\Gamma. \tag{4.1}$$

By the maximum modulus principle then it follows that

$$\|f_1\|_{\Phi_1(\partial D_1)} \le C \log n \|R_n\|_\Gamma \tag{4.2}$$

and

$$\|f_2\|_{\Phi_2(\partial D_2)} \le C \log n \|R_n\|_\Gamma. \tag{4.3}$$

Set $F_1 := f_1(\Phi_1)$ and $F_2 := f_2(\Phi_2)$. These are meromorphic functions in D_1 and D_2 resp. with poles at $\alpha_j' := \Phi_1^{-1}(\alpha_j)$, $j = 1, \ldots, N_1$ and at $\beta_k' := \Phi_2^{-1}(\beta_k)$, $k = 1, \ldots, N_2$.

Let $F_1 = F_{1,r} + F_{1,a}$ be the decomposition of F_1 with respect to the unit disk into rational and analytic parts with $F_{1,r}(\infty) = 0$, and in a similar fashion, let $F_2 = F_{2,r} + F_{2,a}$ be the decomposition of F_2 with respect to the exterior of the unit disk into rational and analytic parts with $F_{2,r}(0) = 0$. (Here r refers to the rational part, a refers to the analytic part.) Hence, we have by Proposition 3.6

$$\|F_{j,r}\|_\mathbf{T}, \|F_{j,a}\|_\mathbf{T} \le C \log n \|F_j\|_\mathbf{T}, \qquad j = 1, 2.$$

Thus, $F_{1,r}$ is a rational function with poles at $\alpha_j' \in \mathbf{D}$, so by the maximum modulus theorem and (4.1) (applied to $\mathbf{T}$ rather than to Γ) we have

$$\|F_{1,r}\|_{\mathbf{T}(r_1)} \le C \log n \|F_1\|_\mathbf{T} \le C \log^2 n \|R_n\|_\Gamma, \tag{4.4}$$

where we used that $\|F_1\|_{\mathbf{T}} = \|f_1\|_\Gamma$. But (4.2) is the same as

$$\|F_1\|_{\mathbf{T}(r_1)} \le C \log n \|R_n\|_\Gamma,$$

so we can conclude also

$$\|F_{1,a}\|_{\mathbf{T}(r_1)} \le C \log^2 n \, \|R_n\|_\Gamma \,. \tag{4.5}$$

Thus, $F_{1,a}$ is an analytic function in D_1 with the bound in (4.5). Apply now Proposition 3.7 to this function and to the unit circle as K (and with a somewhat larger concentric circle as τ) with degree $m = [\sqrt{n}]$. According to that proposition there are $C, c > 0$ and polynomials $S_1 = S_{1,\sqrt{n}}$ of degree at most $\sqrt{n}$ such that

$$\|F_{1,a} - S_1\|_{\mathbf{T}} \le Ce^{-c\sqrt{n}}\|R_n\|_\Gamma, \quad S_1(1) = F_{1,a}(1),\ S_1'(1) = F_{1,a}'(1).$$

Therefore, $\tilde{\mathcal{R}}_1 := F_{1,r} + S_1$ is a rational function with poles at α_j', $j = 1, \ldots, N_1$ and with a pole at ∞ with order at most $\sqrt{n}$ which satisfies

$$\left\|F_1 - \tilde{\mathcal{R}}_1\right\|_{\mathbf{T}} \le Ce^{-c\sqrt{n}}\|R_n\|_\Gamma, \quad \tilde{\mathcal{R}}_1(1) = F_1(1),\ \tilde{\mathcal{R}}_1'(1) = F_1'(1) \tag{4.6}$$

In a similar vein, if we consider $F_2(1/v)$ and use (4.3), then we get a polynomial S_2 of degree at most $\sqrt{n}$ such that

$$\|F_{2,a}(1/v) - S_2(v)\|_{\mathbf{T}} \le Ce^{-c\sqrt{n}}\|R_n\|_\Gamma, \quad S_2(1) = F_{2,a}(1),\ S_2'(1) = -F_{2,a}'(1)$$

But then $\tilde{\mathcal{R}}_2(v) := F_{2,r}(v) + S_2(1/v)$ is a rational function with poles at β_k', $k = 1, \ldots, N_2$ and with a pole at 0 of order at most $\sqrt{n}$ that satisfies

$$\left\|F_2 - \tilde{\mathcal{R}}_2\right\|_{\mathbf{T}} \le Ce^{-c\sqrt{n}}\|R_n\|_\Gamma, \quad \tilde{\mathcal{R}}_2(1) = F_2(1),\ \tilde{\mathcal{R}}_2'(1) = F_2'(1). \tag{4.7}$$

What we have obtained is that the rational function $\tilde{\mathcal{R}} := \tilde{\mathcal{R}}_1 + \tilde{\mathcal{R}}_2$ is of distance $\le Ce^{-c\sqrt{n}}\|R_n\|_\Gamma$ from $F_1 + F_2$ on the unit circle and it satisfies

$$\tilde{\mathcal{R}}(1) = (F_1 + F_2)(1) = f_1(z_0) + f_2(z_0) = R_n(z_0) \tag{4.8}$$

and using (3.2),

$$\tilde{\mathcal{R}}'(1) = (F_1' + F_2')(1) = f_1'(z_0)\Phi_1'(1) + f_2'(z_0)\Phi_2'(1) = R_n'(z_0)\Phi_1'(1). \tag{4.9}$$

Consider now $F_1 + F_2$ on the unit circle, i.e.

$$F_1(e^{it}) + F_2(e^{it}) = f_1(\Phi_2(e^{it})) + f_2(\Phi_2(e^{it})) + f_1(\Phi_1(e^{it})) - f_1(\Phi_2(e^{it})).$$

The sum of the first two terms on the right is $R_n(\Phi_2(e^{it}))$, and this is at most $\|R_n\|_\Gamma$ in absolute value. Next, we estimate the difference of the last two terms.

The function $\Phi_1(v)-\Phi_2(v)$ is analytic in the ring $r_2<|v|<r_1$ and it is bounded there with a bound depending only on Γ, r_1, r_2, furthermore it has a double zero at $v=1$ (because of (3.2)). These imply

$$|\Phi_1(e^{it})-\Phi_2(e^{it})|\le C|e^{it}-1|^2\le Ct^2, \qquad t\in[-\pi,\pi],$$

with some constant C. By Proposition 3.1 we have with (4.1) also the bound

$$\|f_1'\|_\Gamma \le Cn\log n\|R_n\|_\Gamma,$$

and these last two facts give us (just integrate f_1' along the shorter arc of Γ in between $\Phi_1(e^{it})$ and $\Phi_2(e^{it})$ and use that the length of this arc is at most $C|\Phi_1(e^{it})-\Phi_2(e^{it})|$)

$$|f_1(\Phi_1(e^{it}))-f_1(\Phi_2(e^{it}))|\le Ct^2 n\log n\|R_n\|_\Gamma.$$

By [23, Theorem 4.1] there are polynomials Q of degree at most $[n^{4/5}]$ such that $Q(1)=1$, $\|Q\|_{\mathbf{T}}\le 1$, and with some constants $c_0, C_0>0$

$$|Q(v)|\le C_0\exp(-c_0 n^{4/5}|v-1|^{3/2}), \qquad |v|=1.$$

With this Q consider the rational function $\mathcal{R}(v)=\tilde{\mathcal{R}}(v)Q(v)$. On the unit circle this is closer than $Ce^{-c\sqrt{n}}\|R_n\|_\Gamma$ to $(F_1+F_2)Q$, and in view of what we have just proven, we have at $v=e^{it}$

$$|(F_1(v)+F_2(v))Q(v)|\le \|R_n\|_\Gamma + Ct^2 n\log n C_0\exp\left(-c_0 n^{4/5}|t/2|^{3/2}\right)\|R_n\|_\Gamma.$$

On the right

$$\begin{aligned} &t^2 n\log n\exp\left(-c_0 n^{4/5}|t/2|^{3/2}\right)\\ &\qquad = 4\left(n^{4/5}|t/2|^{3/2}\right)^{4/3}\exp\left(-c_0 n^{4/5}|t/2|^{3/2}\right)\frac{\log n}{n^{1/15}}\le C\frac{\log n}{n^{1/15}} \end{aligned}$$

because $|x|^{4/3}\exp(-c_0|x|)$ is bounded on the real line.

All in all, we obtain

$$\|\mathcal{R}\|_{\mathbf{T}}\le(1+o(1))\|R_n\|_\Gamma, \tag{4.10}$$

and

$$\begin{aligned} |\mathcal{R}'(1)| = |\tilde{\mathcal{R}}'(1)Q(1)+\tilde{\mathcal{R}}(1)Q'(1)| = |\tilde{\mathcal{R}}'(1)| + O\left(|\tilde{\mathcal{R}}(1)||Q'(1)|\right)\\ = |R_n'(z_0)| + O(n^{4/5})\|R_n\|_\Gamma, \end{aligned}$$

where we used $Q(1)=1$, (4.8)–(4.9), $|\Phi_1'(1)|=1$ and the classical Bernstein inequality for $Q'(1)$, which gives the bound $n^{4/5}$ for the derivative of Q.

The poles of $\mathcal{R}$ are at α_j', $1\le j\le N_1$, and at β_k', $1\le k\le N_2$, as well as a $\le n^{1/2}$ order pole at 0 (coming from the construction of $S_{2,n}$) and a $\le n^{1/2}+n^{4/5}$ order pole at ∞ (coming from the construction of $S_{1,n}$ and the use of Q).

Now we apply the Borwein-Erdélyi inequality (Proposition 3.5) to $|\mathcal{R}'(1)|$ to obtain

$$\begin{aligned}
|R_n'(z_0)| &\leq |\mathcal{R}'(1)| + O(n^{4/5})\|R_n\|_\Gamma \\
&\leq \|\mathcal{R}\|_{\mathbf{T}} \max\left(\sum_k \frac{\partial g_{\mathbf{D}_+}(1,\beta_k')}{\partial \mathbf{n}_+} + (n^{1/2}+n^{4/5})\frac{\partial g_{\mathbf{D}_+}(1,\infty)}{\partial \mathbf{n}_+},\right. \\
&\qquad \left.\sum_j \frac{\partial g_{\mathbf{D}}(1,\alpha_j')}{\partial \mathbf{n}_-} + n^{1/2}\frac{\partial g_{\mathbf{D}}(1,0)}{\partial \mathbf{n}_-}\right) + O(n^{4/5})\|R_n\|_\Gamma.
\end{aligned}$$

If we use here how the normal derivatives transform under the mappings Φ_1 and Φ_2 as in Proposition 3.3, then we get from (4.10)

$$\begin{aligned}
|R_n'(z_0)| \leq\ & (1+o(1))\|R_n\|_\Gamma \max\left(\sum_{a\in Z\cap G_+} \frac{\partial g_{G_+}(z_0,a)}{\partial \mathbf{n}_+} + (n^{1/2}+n^{4/5})\frac{\partial g_{G_+}(z_0,\infty)}{\partial \mathbf{n}_+},\right. \\
&\left.\sum_{a\in Z\cap G_-} \frac{\partial g_{G_-}(z_0,a)}{\partial \mathbf{n}_-} + n^{1/2}\frac{\partial g_{G_-}(z_0,\Phi_1(0))}{\partial \mathbf{n}_-}\right) + O(n^{4/5})\|R_n\|_\Gamma.
\end{aligned}$$

Since, by (3.7)–(3.8), the normal derivatives on the right lie in between two positive constants that depend only on Γ and Z, (2.1) follows (note that one of the sums $\sum_{a\in Z\cap G_+}$ or $\sum_{a\in Z\cap G_-}$ contains at least $n/2$ terms).

■

5 The Bernstein-type inequality on analytic arcs

In this section we prove Theorem 2.4 in the case when the arc Γ is analytic. We shall reduce this case to Theorem 2.1 for analytic Jordan curves that has been proven in the preceding section. We shall use the Joukowskii map to transform the arc setting to the curve setting.

For clearer notation let us write for the arc in Theorem 2.4 Γ_0. We may assume that the endpoints of Γ_0 are ± 1. Consider the pre-image Γ of Γ_0 under the Joukowskii map $z = F(u) = (u + 1/u)/2$. Then Γ is a Jordan curve, and if $G_\pm$ denote the inner and outer domains to Γ, then F is a conformal map from both G_- and from G_+ onto $\overline{\mathbf{C}} \setminus \Gamma_0$. Furthermore, the analyticity of Γ_0 implies that Γ is an analytic Jordan curve, see [11, Proposition 5].

Denote the inverse of $z = F(u)$ restricted to G_- by $F_1^{-1}(z) = u$ and that restricted to G_+ by $F_2^{-1}(z) = u$. So $F_j(z) = z \pm \sqrt{z^2-1}$ with an appropriate branch of $\sqrt{z^2-1}$ on the plane cut along Γ_0.

We need the mapping properties of F regarding normal vectors, for full details, we refer to [11] p. 879. Briefly, for any $z_0 \in \Gamma_0$ that is not one of the endpoints of Γ_0 there are exactly two $u_1, u_2 \in \Gamma$, $u_1 \neq u_2$ such that $F(u_1) = F(u_2) = z_0$. Denote the normal vectors to Γ pointing outward by $\mathbf{n}_+$ and the normal vectors pointing inward by $\mathbf{n}_-$ (it is usually unambiguous from the

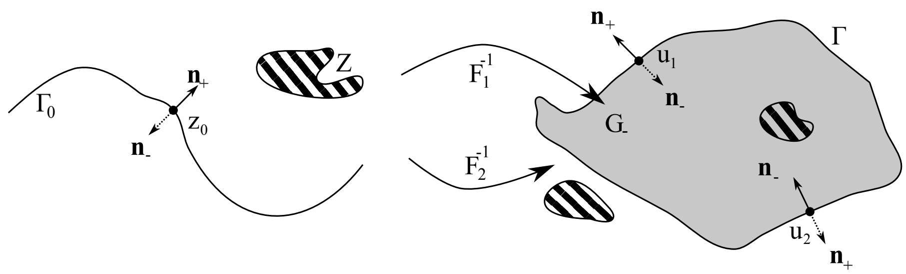

Figure 2: The open-up

context at which point $u \in \Gamma$ we are referring to). By reindexing u_1 and u_2 (and possibly reversing the parametrization of Γ_0), we may assume that the (direction of the) normal vector $\mathbf{n}_+(u_1)$ is mapped by F to the (direction of the) normal vector $\mathbf{n}_+(z_0)$. This then implies that (the directions of) $\mathbf{n}_+(u_1)$, $\mathbf{n}_-(u_1)$ and $\mathbf{n}_+(u_2)$, $\mathbf{n}_-(u_2)$ are mapped by F to (the directions of) $\mathbf{n}_+$, $\mathbf{n}_-$, $\mathbf{n}_-$, $\mathbf{n}_+$ at z_0, respectively. These mappings are depicted on Figure 2.

The corresponding normal derivatives of the Green's functions are related as follows.

Proposition 5.1 *We have for $a \in \overline{\mathbf{C}} \setminus \Gamma$*

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_0}(z_0,a)}{\partial \mathbf{n}_-} = \frac{\partial g_{G_-}\left(u_1, F_1^{-1}(a)\right)}{\partial \mathbf{n}_-} / \left|F'(u_1)\right| = \frac{\partial g_{G_+}\left(u_2, F_2^{-1}(a)\right)}{\partial \mathbf{n}_+} / \left|F'(u_2)\right|$$

and, similarly for the other side,

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_0}(z_0,a)}{\partial \mathbf{n}_+} = \frac{\partial g_{G_-}\left(u_2, F_1^{-1}(a)\right)}{\partial \mathbf{n}_-} / \left|F'(u_2)\right| = \frac{\partial g_{G_+}\left(u_1, F_2^{-1}(a)\right)}{\partial \mathbf{n}_+} / \left|F'(u_1)\right|.$$

This proposition follows immediately from [11, Proposition 6] and is a two-to-one mapping analogue of Proposition 3.3.

After these preliminaries let us turn to the proof of (2.4) at a point $z_0 \in \Gamma_0$. Consider $f_1(u) := R_n(F(u))$ on the analytic Jordan curve Γ at u_1 (where $F(u_1) = z_0$). This is a rational function with poles at $F_1^{-1}(a) \in G_-$ and at $F_2^{-1}(a) \in G_+$, where a runs through the poles of R_n. According to (2.1) (that has been verified in Section 4 for analytic curves) we have

$$|f_1'(u_1)| \le (1+o(1))\,\|f_1\|_\Gamma$$
$$\cdot \max\left(\sum_a \frac{\partial g_{G_-}\left(u_1, F_1^{-1}(a)\right)}{\partial \mathbf{n}_-}, \sum_a \frac{\partial g_{G_+}\left(u_1, F_2^{-1}(a)\right)}{\partial \mathbf{n}_+}\right),$$

where a runs through the poles of R_n (counting multiplicities). If we use here that $\|f_1\|_\Gamma = \|R_n\|_{\Gamma_0}$ and $f_1'(u_1) = R_n'(z_0)\,F'(u_1)$, we get from Proposition 5.1

$$|R_n'(z_0)| \le (1+o(1))\,\|R_n\|_{\Gamma_0}$$
$$\cdot \max\left(\sum_a \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_0}(z_0, a)}{\partial \mathbf{n}_-}, \sum_a \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_0}(z_0, a)}{\partial \mathbf{n}_+}\right),$$

which is (2.4) when Γ is replaced by Γ_0.

■

6 Proof of Theorem 2.4

In this section we verify (2.4) for C^2 arcs. Recall that in Section 5 (2.4) has already been proven for analytic arcs and we shall reduce the C^2 case to that by approximation similar to what was used in [24].

In the proof we shall frequently identify a Jordan arc with its parametric representation.

By assumption, Γ has a twice differentiable parametrization $\gamma(t)$, $t \in [-1,1]$, such that $\gamma'(t) \neq 0$ and γ'' is continuous. We may assume that $z_0 = 0$ and that the real line is tangent to Γ at 0, and also that $\gamma(0) = 0$, $\gamma'(0) > 0$. There is an M_1 such that for all $t \in [-1,1]$

$$\frac{1}{M_1} \le |\gamma'(t)| \le M_1, \qquad |\gamma''(t)| \le M_1. \tag{6.1}$$

Let $\gamma_0 := \gamma$, and for some $0 < \tau_0 < 1$ and for all $0 < \tau \le \tau_0$ choose a polynomial g_τ such that

$$|\gamma'' - g_\tau| \le \tau, \tag{6.2}$$

and set

$$\gamma_\tau(t) = \int_0^t \left(\int_0^u g_\tau(v)dv + \gamma_0'(0)\right) du. \tag{6.3}$$

It is clear that

$$|\gamma_\tau(t) - \gamma_0(t)| \le \tau|t|^2, \quad |\gamma_\tau'(t) - \gamma_0'(t)| \le \tau|t|, \quad |\gamma_\tau''(t) - \gamma_0''(t)| \le \tau. \tag{6.4}$$

It was proved in [24, Section 2] that for small τ, say for all $\tau \le \tau_0$ (which can be achieved by decreasing τ_0 if necessary), these γ_τ are analytic Jordan arcs, and

$$g_{\overline{\mathbf{C}}\setminus\gamma_0}(z,\infty) \le M_2\sqrt{\tau}|z|^2, \quad z \in \gamma_\tau, \tag{6.5}$$

with some constant M_2 that is independent of τ and z. We need similar estimates for all $g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(z,a)$, $a \in Z$. To get them consider the closure of the set $\cup_{0\le\tau\le\tau_0}\gamma_\tau$ and its polynomial convex hull

$$K = \mathrm{Pc}\left(\overline{\bigcup_{0\le\tau\le\tau_0}\gamma_\tau}\right),$$

which is the union of that closure with all the bounded components of its complement. Now this is a situation when the results from Section 3.6 can be applied. From Corollary 3.9 and from (6.5) we can conclude for all $a \in Z$

$$g_{\overline{\mathbf{C}}\setminus\gamma_0}(z,a) \le M_3\sqrt{\tau}|z|^2, \quad z \in \gamma_\tau \tag{6.6}$$

with some constant M_3.

Let $\mathbf{n}_\pm$ denote the two normals to γ_τ at the origin. Note that $\mathbf{n}_\pm$ are common to all the arcs γ_τ, $0 \le \tau \le \tau_0$.

Lemma 6.1 *For small τ_0 the normal derivatives*

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(0,a)}{\partial \mathbf{n}_\pm}, \qquad 0 \le \tau \le \tau_0,\ a \in Z \cup \{\infty\},$$

are uniformly bounded from below and above by a positive number.

Proof. It was proven in [24, Appendix 1] that

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(0,\infty)}{\partial \mathbf{n}_\pm} \to \frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_0}(0,\infty)}{\partial \mathbf{n}_\pm} \tag{6.7}$$

as $\tau \to 0$, and the value on the right is positive and finite. Now just invoke Corollary 3.9 (note that (3.6) implies similar inequalities for the normal derivatives).

■

Next we mention that (6.4) implies the following: no matter how $\eta > 0$ is given, there is a $\tau_\eta < \tau_0$ such that for $\tau < \tau_\eta$ we have

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(0,\infty)}{\partial \mathbf{n}_\pm} < (1+\eta)\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_0}(0,\infty)}{\partial \mathbf{n}_\pm}. \tag{6.8}$$

In fact, (6.7) was proven in [24, Appendix 1, (6.1)] under the assumption (6.4), and since the normal derivatives on the right are not zero, (6.8) follows.

We shall also need this inequality when ∞ is replaced by an arbitrary pole $a \in Z$. Let $a \in Z$ be arbitrary, and consider the mapping $\varphi_a(z) = 1/(z-a)$.

Under this mapping γ_τ is mapped into $\varphi_a(\gamma_\tau)$ with parametrization $\varphi_a(\gamma_\tau(t))$, $t \in [-1, 1]$, and it is clear that (6.4) implies its analogue for the image curves:

$$|\varphi_a(\gamma_\tau)(t) - \varphi_a(\gamma_0)(t)| \le C\tau|t|^2, \qquad \begin{array}{l} |(\varphi_a(\gamma_\tau))'(t) - (\varphi_a(\gamma_0))'(t)| \le C\tau|t|, \\ |(\varphi_a(\gamma_\tau))''(t) - (\varphi_a(\gamma_0))''(t)| \le C\tau, \end{array}$$

with some constant C that is independent of τ and $a \in Z$. Furthermore,

$$g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(z, a) = g_{\overline{\mathbf{C}}\setminus\varphi_a(\gamma_\tau)}(\varphi_a(z), \infty),$$

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(0, a)}{\partial \mathbf{n}_\pm} = \frac{\partial g_{\overline{\mathbf{C}}\setminus\varphi_a(\gamma_\tau)}(\varphi_a(0), \infty)}{\partial \mathbf{n}(\varphi_a(0))_\pm}|\varphi_a'(0)|.$$

Now if we use these in the proof of [24, Appendix 1] and use also Lemma 6.1, then we obtain that for every $\eta > 0$ there is a $\tau_\eta < \tau_0$ such that for $\tau < \tau_\eta$ and $a \in Z$ we have

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(0, a)}{\partial \mathbf{n}_\pm} < (1 + \eta)\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_0}(0, a)}{\partial \mathbf{n}_\pm}. \tag{6.9}$$

An inspection of the proof reveals that τ_η can be made independent of $a \in Z$, so (6.9) is uniform in $a \in Z$.

After these preparations let R_n be a rational function with poles in Z such that total order of its poles (including possibly the pole at ∞) is n. We use

$$|R_n(z)| \le \exp\left(\sum_a g_{\overline{\mathbf{C}}\setminus\Gamma}(z, a)\right)\|R_n\|_\Gamma, \tag{6.10}$$

where the summation is for all poles of R_n. This is the analogue of (3.1), and its proof is the same that we gave for (3.1). Hence, in view of (6.6), we have for $z \in \gamma_\tau$ (recall that $\gamma_0 = \Gamma$)

$$|R_n(z)| \le \|R_n\|_\Gamma \exp\Big(nM_3\sqrt{\tau}|z|^2\Big). \tag{6.11}$$

The polynomial convex hull K introduced above has the property that there is a disk (say in the upper half plane) in the complement of K which contains the point 0 on its boundary. Indeed, this easily follows from the construction of the curves γ_τ. Now we use [23, Theorem 4.1], according to which there are constants c_1, C_1 and for each m polynomials Q_m of degree at most m such that

(i) $Q_m(0) = 1$,

(ii) $|Q_m(z)| \le 1, \quad z \in K,$ (6.12)

(iii) $|Q_m(z)| \le C_1 e^{-c_1 m|z|^2}, \quad z \in K.$

For some small $\varepsilon > 0$ consider $R_n(z)Q_{\varepsilon n}(z)$. This is a rational function with poles in Z and at ∞, and it will be important that the pole at infinity coming from $Q_{\varepsilon n}$ is of order at most εn. We estimate this product on γ_τ as follows. Let

$z \in \gamma_\tau$ and let $0 < \eta < 1$ be given. If $|z| \le \sqrt{2\log(C_1)/c_1\varepsilon n}$, then (6.11) and (ii) yield

$$|R_n(z)Q_{\varepsilon n}(z)| \le \exp\Big(M_3\sqrt{\tau}2\log(C_1)/c_1\varepsilon\Big)\|R_n\|_\Gamma,$$

and the right hand side is smaller than $(1+\eta)\|R_n\|_\Gamma$ if $\tau < (\eta c_1\varepsilon/4M_3\log C_1)^2$. On the other hand, if $|z| > \sqrt{2\log(C_1)/c_1\varepsilon n}$, then (6.11) and (iii) give

$$|R_n(z)Q_{\varepsilon n}(z)| \le \|R_n\|_\Gamma C_1 \exp\Big(nM_3\sqrt{\tau}|z|^2 - c_1\varepsilon n|z|^2\Big). \tag{6.13}$$

For $\sqrt{\tau} < c_1\varepsilon/2M_3$ the exponent is at most

$$-n(c_1/2)\varepsilon|z|^2 \le \log(1/C_1)$$

so in this case we have

$$|R_n(z)Q_{\varepsilon n}(z)| \le \|R_n\|_\Gamma. \tag{6.14}$$

What we have shown is that

$$\|R_nQ_{\varepsilon n}\|_{\gamma_\tau} \le (1+\eta)\|R_n\|_\Gamma \tag{6.15}$$

if τ is small, say $\tau < \tau_\eta^*$. Fix such a τ. The corresponding γ_τ is an analytic arc, so we can apply (2.4) to it and to the rational function $R_nQ_{\varepsilon n}$ (recall that (2.4) has already been proven for analytic arcs in Section 5). It follows that

$$|(R_nQ_{\varepsilon n})'(0)| \le (1+o(1))\|R_nQ_{\varepsilon n}\|_{\gamma_\tau} \max\left(\sum_a{}' \frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(0,a)}{\partial \mathbf{n}_+}, \sum_a{}' \frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(0,a)}{\partial \mathbf{n}_-}\right), \tag{6.16}$$

where now $\sum_a'$ means that the summation is for the poles of $R_nQ_{\varepsilon n}$, i.e. for the poles of R_n as well as for the at most εn poles $a = \infty$ that possibly come from $Q_{\varepsilon n}$. Note that some of the poles may be cancelled in $R_nQ_{\varepsilon n}$, but the inequality

$$\sum_a{}' \frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(0,a)}{\partial \mathbf{n}_\pm} \le \sum_a \frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(0,a)}{\partial \mathbf{n}_\pm} + \varepsilon n \frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(0,\infty)}{\partial \mathbf{n}_\pm}, \tag{6.17}$$

(where on the right the summation is only on the original poles of R_n) holds in that case, as well. For the first sum on the right we use (6.9) and for the second sum Lemma 6.1 to conclude

$$\sum_a{}' \frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(0,a)}{\partial \mathbf{n}_\pm} \le (1+\eta)\sum_a \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(0,a)}{\partial \mathbf{n}_\pm} + C_2\varepsilon n \tag{6.18}$$

with some C_2 that depends only on Γ. Since the sum on the right of (6.18) is $\ge c_2 n$ with some fixed $c_2 > 0$ again by Lemma 6.1, we obtain from (6.15) and (6.16)

$$\begin{aligned}|(R_nQ_{\varepsilon n})'(0)| \quad &\le \quad (1+o(1))(1+\eta)^2\|R_n\|_\Gamma(1+C_2\varepsilon/c_2) \times \\ & \max\left(\sum_a \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(0,a)}{\partial \mathbf{n}_+}, \sum_a \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(0,a)}{\partial \mathbf{n}_-}\right).\end{aligned}$$

In view of $Q_{\varepsilon n}(0) = 1$, on the left

$$(R_n Q_{\varepsilon n})'(0) = R_n'(0) + R_n(0) Q_{\varepsilon n}'(0),$$

and for the second term we get again from (2.4) (known for the analytic arc γ_τ by Section 5) and from $\|Q_{\varepsilon n}\|_{\gamma_\tau} \le 1$

$$|R_n(0) Q_{\varepsilon n}'(0)| \le (1 + o(1)) \|R_n\|_\Gamma n\varepsilon \max\left(\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(0,\infty)}{\partial \mathbf{n}_+}, \frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma_\tau}(0,\infty)}{\partial \mathbf{n}_-} \right),$$

and we can again apply (6.8) to the right hand side. If we use again Lemma 6.1 as before, we finally obtain

$$|R_n'(0)| \le (1+o(1))(1+\eta)^2(1+C_3\varepsilon)\|R_n\|_\Gamma \max\left(\sum_a \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(0,a)}{\partial \mathbf{n}_+}, \sum_a \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(0,a)}{\partial \mathbf{n}_-} \right)$$

with some constant C_3 independent of ε and η. Now this is true for all $\varepsilon, \eta > 0$ so the claim (2.4) follows.

We shall not prove the last statement concerning the uniformity of the estimate, for the argument is very similar to the one given in the proof of [24, Theorem 1].

■

7 Proof of Theorem 2.1

In this section we prove the inequality (2.1) for C^2 curves. Recall that in Section 4 the inequality (2.1) has already been proven for analytic curves, which was the basis of all subsequent results. In the present section we show how (2.1) for C^2 curves can be deduced from the inequality (2.4) for C^2 arcs.

Thus, let Γ be a positively oriented C^2 smooth Jordan curve and z_0 a point on Γ. Let $w_0 \ne z_0$ be another point of Γ (think of w_0 as lying "far" from z_0), and for $m = 1, 2, \ldots$ let $w_m \in \Gamma$ be the point on Γ such that the arc $\overline{w_0 w_m}$ (in the orientation of Γ) is of length $1/m$. Such a w_m exists and the arc $\overline{w_0 w_m}$ does not contain z_0 for all sufficiently large m, say for $m \ge m_0$. Remove now the (open) arc $\overline{w_0 w_m}$ from Γ to get the Jordan arc $\Gamma_m = \Gamma \setminus \overline{w_0 w_m}$. We can apply (2.4) to this Γ_m, and what we are going to show is that the so obtained inequality proves (2.1) as $m \to \infty$.

Let $a \in G_- \cap Z$. We show that, as $m \to \infty$,

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z_0,a)}{\partial \mathbf{n}_-} \to \frac{\partial g_{G_-}(z_0,a)}{\partial \mathbf{n}_-} \tag{7.1}$$

and

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z_0,a)}{\partial \mathbf{n}_+} \to 0, \tag{7.2}$$

uniformly in $a \in G_- \cap Z$. Indeed, since $\Gamma_m \subset \Gamma_{m+1}$, the Green's functions $g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z,a)$ decrease as m increases. Furthermore, $g_{\overline{\mathbf{C}}\setminus\Gamma_{m_0}}(z,a)$ is continuous at w_0, so for every $\varepsilon > 0$ there is an m_ε such that for $z \in \overline{w_0 w_{m_\varepsilon}}$ we have $g_{\overline{\mathbf{C}}\setminus\Gamma_{m_0}}(z,a) < \varepsilon$. In view of Corollary 3.9 this m_ε can be the same for all $a \in Z \cap G_-$ since the Green's functions $g_{\overline{\mathbf{C}}\setminus\Gamma_{m_0}}(z,a)$ with respect to different $a \in Z \cap G_-$ are comparable inside a Jordan curve σ that encloses Γ_{m_0}. This then implies for $m \geq m_\varepsilon$ and $z \in \overline{w_0 w_m}$

$$0 < g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z,a) \leq g_{\overline{\mathbf{C}}\setminus\Gamma_{m_0}}(z,a) < \varepsilon. \tag{7.3}$$

Thus, for $m \geq m_\varepsilon$ the function $g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z,a) - g_{G_-}(z,a)$ is positive and harmonic in G_-, and on the boundary of G_- it is either 0 or $< \varepsilon$, so by the maximum principle it is $< \varepsilon$ everywhere in the closure $\overline{G_-}$. Let now $a_0 \in G_+$ be fixed, i.e. a_0 lies in the outer domain to Γ, and let $I \subset \Gamma$ be a subarc of Γ which does not contain z_0 and which contains $\overline{w_0 w_{m_0}}$ in its interior, and set $\Gamma_I = \Gamma \setminus I$. Then $g_{\overline{\mathbf{C}}\setminus\Gamma_I}(z,a_0)$ has a strictly positive lower bound c_0 on $\overline{w_0 w_{m_0}}$ (note that this arc lies inside the domain $\overline{\mathbf{C}} \setminus \Gamma_I$), therefore, in view of (7.3), we have

$$0 < g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z,a) - g_{G_-}(z,a) < \frac{\varepsilon}{c_0} g_{\overline{\mathbf{C}}\setminus\Gamma_I}(z,a_0) \tag{7.4}$$

on the boundary of G_- provided $m \geq m_\varepsilon$. By the maximum principle this inequality then holds throughout G_- (note that both sides are harmonic there), and hence we have for $m \geq m_\varepsilon$

$$0 \leq \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z_0,a)}{\partial \mathbf{n}_-} - \frac{\partial g_{G_-}(z_0,a)}{\partial \mathbf{n}_-} \leq \frac{\varepsilon}{c_0} \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_I}(z_0,a_0)}{\partial \mathbf{n}_-}, \tag{7.5}$$

and upon letting $\varepsilon \to 0$ we obtain (7.1).

The proof of (7.2) is much the same, just work now in the exterior domain G_+, and use the reference Green's function $g_{\overline{\mathbf{C}}\setminus\Gamma_I}(z,b_0)$ with b_0 lying in the bounded domain G_-. In this case $g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z,a)$ is harmonic in G_+ for $a \in G_- \cap Z$ and (7.4) takes the form

$$0 < g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z,a) < \frac{\varepsilon}{c_0} g_{\overline{\mathbf{C}}\setminus\Gamma_I}(z,b_0),$$

from where the conclusion (7.2) can be made as before.

For poles a lying outside Γ we have similarly

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z_0,a)}{\partial \mathbf{n}_+} \to \frac{\partial g_{G_+}(z_0,a)}{\partial \mathbf{n}_+} \tag{7.6}$$

and

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z_0,a)}{\partial \mathbf{n}_-} \to 0, \tag{7.7}$$

uniformly in $a \in G_+ \cap Z$ as $m \to \infty$.

After these preparations we turn to the proof of (2.1). Choose, for a large m, the Jordan arc Γ_m, and apply (2.4) to this Jordan arc and to the rational function R_n in Theorem 2.1. Since $\|R_n\|_{\Gamma_m} \le \|R_n\|_\Gamma$, we obtain

$$|R_n'(z_0)| \le (1+o(1))\|R_n\|_\Gamma \max\left(\sum_{a\in Z} \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z_0,a)}{\partial \mathbf{n}_+}, \sum_{a\in Z} \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z_0,a)}{\partial \mathbf{n}_-}\right) \tag{7.8}$$

where the $o(1)$ term may depend on m. In view of (7.1)–(7.2) and (7.6)–(7.7) (use also (3.7) and (3.8))

$$\sum_{a\in Z} \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z_0,a)}{\partial \mathbf{n}_+} \le (1+o_m(1)) \sum_{a\in Z\cap G_+} \frac{\partial g_{G_+}(z_0,a)}{\partial \mathbf{n}_+} + o_m(1)n$$

and

$$\sum_{a\in Z} \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_m}(z_0,a)}{\partial \mathbf{n}_-} \le (1+o_m(1)) \sum_{a\in Z\cap G_-} \frac{\partial g_{G_-}(z_0,a)}{\partial \mathbf{n}_-} + o_m(1)n,$$

where $o_m(1)$ denotes a quantity that tends to 0 as $m\to\infty$. These imply that the maximum on the right of (7.8) is at most

$$(1+o_m(1)) \max\left(\sum_{a\in Z\cap G_+} \frac{\partial g_{G_+}(z_0,a)}{\partial \mathbf{n}_+} + o_m(1)n, \sum_{a\in Z\cap G_-} \frac{\partial g_{G_-}(z_0,a)}{\partial \mathbf{n}_-} + o_m(1)n\right),$$

which is

$$(1+o_m(1)) \max\left(\sum_{a\in Z\cap G_+} \frac{\partial g_{G_+}(z_0,a)}{\partial \mathbf{n}_+}, \sum_{a\in Z\cap G_-} \frac{\partial g_{G_-}(z_0,a)}{\partial \mathbf{n}_-}\right)$$

because of (3.7)–(3.8). Therefore, we obtain (2.1) from (7.8) by letting $n\to\infty$ and at the same time $m\to\infty$ very slowly.

A routine check shows that the proof runs uniformly in $z_0\in\Gamma$ lying on any proper arc J of Γ. In fact, the proof gives that uniformity provided the normal derivative on the right of (7.5) lies in between two positive constants independently of $z_0\in J$, which can be easily proven using the method of Proposition 3.10 (which was based on the Kellogg-Warschawski theorem and that is uniform in z_0 in the given range). From here the uniformity of (2.1) in $z_0\in\Gamma$ follows by considering two such arcs J that together cover Γ.

■

8 Proof of (2.12)

In the proof of Theorem 2.8 we shall need (2.12) which we verify in this section. The proof uses induction on k, the $k=1$ case is covered by Theorem 2.4.

Let R_n and J as in (2.12). First of all we remark that by [23, Theorem 7.1], $g_{\overline{\mathbf{C}}\setminus\Gamma}(z,\infty)$ is Hölder 1/2 continuous: for all $z \in \overline{\mathbf{C}}$

$$g_{\overline{\mathbf{C}}\setminus\Gamma}(z,\infty) \le M\mathrm{dist}(z,\Gamma)^{1/2}$$

with some constant M. This combined with Corollary 3.9 (just apply it to $\gamma_0 = \Gamma$) shows that all $g_{\overline{\mathbf{C}}\setminus\Gamma}(z,a)$, $a \in Z$, are uniformly Hölder 1/2 equi-continuous:

$$g_{\overline{\mathbf{C}}\setminus\Gamma}(z,a) \le M_1\mathrm{dist}(z,\Gamma)^{1/2}, \qquad a \in Z,\ \mathrm{dist}(z,\Gamma) \le d,$$

with some constants M_1 and $d > 0$. If we use also (6.10), then we obtain

$$|R_n(z)| \le \|R_n\|_\Gamma \exp\left(nM_1\mathrm{dist}(z,\Gamma)^{1/2}\right), \qquad \mathrm{dist}(z,\Gamma) \le d.$$

In particular, if $z_0 \in \Gamma$ and $C_{1/n^2}(z_0)$ is the circle about z_0 of radius $1/n^2$, then for all $z \in C_{1/n^2}(z_0)$ we have $|R_n(z)| \le \|R_n\|_\Gamma \exp(M_1)$. Thus, Cauchy's integral formula for the k-th derivative at z_0 (written as a contour integral over $C_{1/n^2}(z_0)$) gives for large n

$$|R_n^{(k)}(z_0)| \le k!n^{2k}e^{M_1}\|R_n\|_\Gamma,$$

and since this is true uniformly for all $z_0 \in \Gamma$,

$$\|R_n^{(k)}\|_\Gamma \le C_k n^{2k}\|R_n\|_\Gamma \tag{8.1}$$

follows with some C_k.

Let

$$V(u) = \max\left(\sum_{i=0}^m n_i \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(u,a_i)}{\partial \mathbf{n}_+}, \sum_{i=0}^m n_i \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(u,a_i)}{\partial \mathbf{n}_-}\right).$$

We shall need the following equi-continuity property of these $V(u)$:

$$V(v) \le (1+\varepsilon)V(z_0) \qquad \text{if } z_0 \in J \text{ and } |v-z_0| < \delta,\ v \in \Gamma, \tag{8.2}$$

with some ε that tends to 0 as $\delta \to 0$. It is clear that this follows if we prove the continuity for each term in $V(u)$, for example, if we show that

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(v,a)}{\partial \mathbf{n}_-} \le (1+\varepsilon)\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z_0,a)}{\partial \mathbf{n}_-} \tag{8.3}$$

if $z_0 \in J$ and $|v - z_0| < \delta$ where ε tends to 0 as $\delta \to 0$. If φ is a conformal map from the unit disk onto $\overline{\mathbf{C}} \setminus \Gamma$ that maps 0 into a, then, just as in (3.9), we have

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(v,a)}{\partial \mathbf{n}_-} = \frac{1}{|\varphi'(\varphi^{-1}(v)|}, \tag{8.4}$$

with the understanding that of the two pre-images $\varphi^{-1}(v)$ of v, in this formula we select the one that is mapped to the left side of Γ by φ. A relatively simple localization (just open up the arc Γ to a C^2 Jordan curve as in Section 5) of the Kellogg-Warschawski theorem ([17, Theorem 3.6]) shows that φ' is positive and continuous away from the pre-images of the endpoints of Γ. This implies (8.3) in view of (8.4).

Suppose now that the claim in (2.12) is true for a k and for all subarcs $J \subset \Gamma$ that do not contain either of the endpoints of Γ. For such a subarc select a subarc $J \subset J^*$ such that J^* has no common endpoint either with J or with Γ. For a $z_0 \in J$ let $Q(v) = Q_{n^{1/3},z_0}(v)$ be as in (i)–(iii) of (6.12) with 0 replaced by z_0 and K replaced by Γ. So this is a polynomial of degree at most $n^{1/3}$ such that $Q(z_0) = 1$, $\|Q\|_\Gamma \le 1$ and if $v \in \Gamma$, then

$$|Q(v)| \le C_1 e^{-c_1 n^{1/3}|v-z_0|^2}. \tag{8.5}$$

Because of the uniform C^2 property of Γ relatively simple consideration shows that here the constants C_1, c_1 are independent of $z_0 \in J$.

Consider any $\delta > 0$ such that the intersection of Γ with the δ-neighborhood of J is part of J^*, and set $f_{k,n,z_0}(v) = R_n^{(k)}(v)Q(v)$. On Γ for this we have the bound

$$O(n^{2k})\exp(-c_1 n^{1/3}\delta^2)\|R_n\|_\Gamma = o(1)\|R_n\|_\Gamma$$

outside the δ-neighborhood of z_0 (see (8.1) and (8.5)). In the δ-neighborhood of any $z_0 \in J$ we have, by $\|Q\|_\Gamma \le 1$ and by the induction hypothesis applied to R_n and to the arc J^*,

$$\begin{aligned} |f_{k,n,z_0}(v)| &\le (1+o(1))\|R_n\|_\Gamma V(v)^k \\ &\le (1+o(1))(1+\varepsilon)^k\|R_n\|_\Gamma V(z_0)^k, \end{aligned}$$

where $\varepsilon \to 0$ as $\delta \to 0$ in view of (8.2). Therefore, $f_{k,n,z_0}(v)$ is a rational function in v of total degree at most $n + n^{1/3} + mk$ (see below) for which

$$\|f_{k,n,z_0}\|_\Gamma \le (1+o(1))\|R_n\|_\Gamma V(z_0)^k,$$

where $o(1) \to 0$ uniformly as $n \to \infty$. The poles of f_{k,n,z_0} agree with the poles a_i of R_n with a slight modification: for $a_i \ne \infty$ the order of a_i in f_{k,n,z_0} is at most $n_i + k$ (see the form (2.6) of R_n), while for $a_0 = \infty$ the order of a_0 is at most $n_0 - k$ plus at most $n^{1/3}$ coming from Q. Upon applying Theorem 2.4 to the rational function f_{k,n,z_0} we obtain (see also (3.7) and (3.8))

$$\begin{aligned} |f'_{k,n,z_0}(z_0)| \le\ & (1+o(1))\|R_n\|_\Gamma V(z_0)^k \times \\ & \left(V(z_0) + O(mk) + n^{1/3}\max\left(\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z_0,\infty)}{\partial \mathbf{n}_+}, \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z_0,\infty)}{\partial \mathbf{n}_-}\right)\right). \end{aligned}$$

In view of (3.7)–(3.8) $V(z_0)$ is much larger (of size n) than the last two terms on the right (which are together of size $O(n^{1/3})$ if z_0 stays away from the endpoints of Γ), hence it follows that

$$|f'_{k,n,z_0}(z_0)| \le (1+o(1))\|R_n\|_\Gamma V(z_0)^{k+1}. \tag{8.6}$$

Since (recall that $Q(z_0)=1$)

$$f'_{k,n,z_0}(z_0)=R_n^{(k+1)}(z_0)+R_n^{(k)}(z_0)Q'(z_0),$$

and the second term on the right is $O(n^{2/3})O(n^k)\|R_n\|_\Gamma$ by the induction assumption and by (8.1) applied to Q with $k=1$ rather than to R_n, we can conclude (2.12) for $k+1$ from (8.6).

From how we have derived this, it follows that this estimate is uniform in $z_0\in J$.

■

9 The Markov-type inequality for higher derivatives

In this section we prove the first part of Theorem 2.8 (the sharpness will be handled in Section 10). The proof uses the symmetrization technique of [24]. It is sufficient to prove (2.14).

First of all we remark that the limits defining $\Omega_a(A)$ in (2.9) exist and are equal for the choices $\mathbf{n}_\pm$. Indeed, let $\varphi_a(z)=1/(z-a)$ be the fractional linear transformation considered before. Then

$$g_{\overline{\mathbf{C}}\setminus\Gamma}(z,a)=g_{\overline{\mathbf{C}}\setminus\varphi_a(\Gamma)}(\varphi_a(z),\infty),$$

so for a $z\in\Gamma$ we have

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z,a)}{\partial\mathbf{n}_\pm}=\frac{\partial g_{\overline{\mathbf{C}}\setminus\varphi_a(\Gamma)}(\varphi_a(z),\infty)}{\partial\mathbf{n}_\pm}|\varphi_a'(z)|,$$

and it has been verified in the proof of [24, Theorem 2] that, as $w\to\varphi_a(A)$, $w\in\varphi_a(\Gamma)$,

$$\sqrt{|w-\varphi_a(A)|}\frac{\partial g_{\overline{\mathbf{C}}\setminus\varphi_a(\Gamma)}(w,\infty)}{\partial\mathbf{n}_\pm}$$

have equal limits, call them $\Omega_\infty(\varphi_a(\Gamma),\varphi_a(A))$, for both choices of $+$ or $-$. Since, as $z\to A$, $z\in\Gamma$, we have $|\varphi_a(z)-\varphi_a(A)|=(1+o(1))|z-A||\varphi_a'(A)|$, it follows that, indeed, the limits

$$\lim_{z\to A,\ z\in\Gamma}\sqrt{|z-A|}\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z,a)}{\partial\mathbf{n}_\pm}=\Omega_\infty(\varphi_a(\Gamma),\varphi_a(A))\sqrt{|\varphi_a'(A)|}$$

exist and are the same for the $+$ or $-$ choices.

Next, we prove the required inequality at the endpoint A. We may assume that $A=0$. Let

$$\Gamma^*=\{z\,|\,z^2\in\Gamma\}.$$

This is a Jordan arc symmetric with respect to the origin. It is not difficult to prove (see [24, Appendix 2]) that Γ^* has C^2 smoothness.

Let R_n be a rational function of degree at most n of the form (2.6), and set $\mathcal{R}_{2n}(z) = R_n(z^2)$. This is a rational function which has $2n$ poles $\pm\sqrt{a_i}$, where a_i runs through the poles of R_n (here $\pm\sqrt{a_i}$ denote the two possible values of $\sqrt{a_i}$ with the understanding that if $a_0 = \infty$, then both values $\pm\sqrt{a_0}$ is ∞). If we apply (2.12) to Γ^* and to the rational function $\mathcal{R}_{2n}$, then we get

$$|\mathcal{R}_{2n}^{(2k)}(0)| \le (1 + o(1))M^{2k}\|\mathcal{R}_{2n}\|_{\Gamma^*}, \tag{9.1}$$

where

$$M = \max_{\pm} \sum_{i=0}^{m} n_i \left\{ \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma^*}(0, \sqrt{a_i})}{\partial \mathbf{n}_\pm} + \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma^*}(0, -\sqrt{a_i})}{\partial \mathbf{n}_\pm} \right\}. \tag{9.2}$$

For $a \neq \infty$

$$g_{\overline{\mathbf{C}}\setminus\Gamma}(z^2, a) = g_{\overline{\mathbf{C}}\setminus\Gamma^*}(z, \sqrt{a}) + g_{\overline{\mathbf{C}}\setminus\Gamma^*}(z, -\sqrt{a}),$$

hence for $z \neq 0$ we have

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma^*}(z, \sqrt{a})}{\partial \mathbf{n}_\pm(z)} + \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma^*}(z, -\sqrt{a})}{\partial \mathbf{n}_\pm(z)} = \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z^2, a)}{\partial \mathbf{n}_\pm(z^2)}|2z| \tag{9.3}$$

(with possibly replacing $\mathbf{n}_\pm$ by $\mathbf{n}_\mp$ on the right), which implies

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma^*}(0, \sqrt{a})}{\partial \mathbf{n}_\pm} + \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma^*}(0, -\sqrt{a})}{\partial \mathbf{n}_\pm} = 2\lim_{w\to 0} \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(w, a)}{\partial \mathbf{n}_\pm(w)}\sqrt{|w|} = 2\Omega_a(A). \tag{9.4}$$

For $a = \infty$ the corresponding calculation is

$$g_{\overline{\mathbf{C}}\setminus\Gamma^*}(z, \infty) = \frac{1}{2} g_{\overline{\mathbf{C}}\setminus\Gamma}(z^2, \infty), \qquad \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma^*}(z, \infty)}{\partial \mathbf{n}_\pm(z)} = \frac{1}{2}\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(z^2, \infty)}{\partial \mathbf{n}_\pm(z^2)}|2z|,$$

and so

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma^*}(0, \infty)}{\partial \mathbf{n}_\pm} = \lim_{w\to 0} \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(w, \infty)}{\partial \mathbf{n}_\pm(w)}\sqrt{|w|} = \Omega_\infty(A). \tag{9.5}$$

Thus, the M in (9.2) is exactly

$$2\left(\sum_{i=0}^{m} n_i \Omega_{a_i}(A)\right). \tag{9.6}$$

In what follows we shall also need that the quantities $\Omega_{a_i}(A)$ are finite and positive, which is immediate from (9.4) and Lemma 6.1 (this latter applied to $\gamma_\tau = \gamma_0 = \Gamma$).

Now we use Faà di Bruno's formula [8] (cf. [12, Theorem 1.3.2])

$$(S(F(z)))^{(2k)} = \sum_{\nu_j} \frac{(2k)!}{\prod_{j=1}^{2k} \nu_j!(j!)^{\nu_j}} S^{(\nu_1+\cdots+\nu_{2k})}(F(z)) \prod_{j=1}^{2k} \left(F^{(j)}(z)\right)^{\nu_j}, \tag{9.7}$$

where the summation is for all nonnegative integers $\nu_1, \ldots, \nu_{2k}$ for which $\nu_1 + 2\nu_2 + 3\nu_3 + \cdots + 2k\nu_{2k} = 2k$, and where 0^0 is defined to be 1 if it occurs on the right. Apply this with $S = R_n$ and $F(z) = z^2$ at $z = 0$:

$$\begin{aligned}
\mathcal{R}_{2n}^{(2k)}(0) &= (R_n(F(z)))^{(2k)}\Big|_{z=0} \\
&= \sum_{\nu_j} \frac{(2k)!}{\prod_{j=1}^{2k} \nu_j!(j!)^{\nu_j}} R_n^{(\nu_1+\cdots+\nu_{2k})}(0) \prod_{j=1}^{2k} \left(F^{(j)}(0)\right)^{\nu_j} \\
&= \frac{(2k)!}{k!2^k} R_n^{(k)}(0) 2^k
\end{aligned}$$

(use that $F^{(j)}(0) = 0$ unless $j = 2$ and then $F^{(2)}(0) = 2$). Hence, in view of (9.1) we obtain

$$|R_n^{(k)}(0)| \le (1 + o(1)) \frac{2^k}{(2k-1)!!} \left(\sum_{i=0}^{m} n_i \Omega_{a_i}(A) \right)^{2k} \|R_n\|_\Gamma, \tag{9.8}$$

where we also used that $\|\mathcal{R}_{2n}\|_{\Gamma^*} = \|R_n\|_\Gamma$. This proves gives the correct bound for the k-th derivative at the endpoint A.

So far we have verified (9.8), which is the claim (2.14), but only at the endpoint $A = 0$ of the arc Γ. We can reduce the Markov type inequality (2.14) to this special case. To achieve that let us denote $\Omega_a(A)$ for the arc Γ by $\Omega_a(\Gamma, A)$. If $z \in \Gamma$ is close to A, then consider the subarc Γ_z which is the arc of Γ from z to B (recall that B is the other endpoint of Γ different from A), so the endpoints of Γ_z are B and z. It is easy to see that the preceding proof of (9.8) was uniform in the sense that it holds uniformly for all Γ_z, $z \in \Gamma$, $|z - A| \le |B - A|/2$ (see the proofs of Theorem 3 and Appendix 1 in [24]), therefore we obtain (replace in (9.8) A by z)

$$|R_n^{(k)}(z)| \le (1 + o(1)) \frac{2^k}{(2k-1)!!} \left(\sum_{i=0}^{m} n_i \Omega_{a_i}(\Gamma_z, z) \right)^{2k} \|R_n\|_{\Gamma_z}, \tag{9.9}$$

where now the quantity $\Omega_a(\Gamma_z, z)$ must be taken with respect to Γ_z, rather than with respect to Γ. Since on the right

$$\|R_n\|_{\Gamma_z} \le \|R_n\|_\Gamma,$$

all what remains to prove is that

$$\lim_{z \to A,\ z \in \Gamma} \Omega_{a_i}(\Gamma_z, z) \to \Omega_{a_i}(\Gamma, A) \tag{9.10}$$

for each a_i, $i = 0, 1, \ldots, m$, as $z \to A$. Indeed, then we obtain from (9.9) and from the fact that, as has been mentioned before, the $\Omega_{a_i}(A)$ quantities are finite and positive, that for any $\varepsilon > 0$

$$|R_n^{(k)}(z)| \le (1 + \varepsilon) \frac{2^k}{(2k-1)!!} \left(\sum_{i=0}^{m} n_i \Omega_{a_i}(\Gamma_z, z) \right)^{2k} \|R_n\|_\Gamma, \tag{9.11}$$

if $z \in \Gamma$ lies sufficiently close to A, say $|z - A| \le \delta$, and n is sufficiently large. On the other hand, (2.12) shows that $R_n^{(k)}(z) = O(n^k)$ on subsets of Γ lying away from the endpoints A, B, in particular this is true for $z \in U$, $|z - A| \ge \delta$. Now this and (9.11) prove the theorem. So it is enough to prove (9.10).

(9.10) has been verified for $a_i = \infty$ in the proof of [24, Theorem 3]. To get it for other a_i just apply the mapping $\varphi_{a_i}(z) = 1/(z - a_i)$ as before to reduce it to the $a_i = \infty$ special case. The reader can easily fill in the details.

■

10 Proof of the sharpness

In this section we prove Theorems 2.3, 2.6 and the second part of Theorem 2.8.

We shall first give the proof for Theorem 2.3. The proof of Theorem 2.6 can be reduced to Theorem 2.3 by attaching a suitable lemniscate as in the proof of Theorem 2.8, so we skip it (actually, a complete proof will be given as part of the proof in Section 10.2 for rational functions of the form (2.6) with fixed poles). However, the sharpness in Theorem 2.8 requires a different approach which will be given in Section 10.2.

10.1 Proof of Theorem 2.3

The idea is as follows. On the unit circle, we use some special rational functions (products of Blaschke factors) for which the Borwein-Erdélyi inequality (Proposition 3.4) is sharp. Then we transfer that back to Γ and approximate the transformed function with rational functions. In other words, we reverse the reasoning in Section 4 and do the "reconstruction step" in the "opposite direction".

Recall that $\mathbf{D} = \{v \mid |v| < 1\}$ and $\mathbf{D}_+ = \{v \mid |v| > 1\} \cup \{\infty\}$, and denote by $B(a, v) = \frac{1-\bar{a}v}{v-a}$ the (reciprocal) Blaschke factor with pole at a.

First, we state cases when we have equality in Proposition 3.4.

Proposition 10.1 *Suppose h is a (reciprocal) Blaschke product with all poles either inside or outside the unit circle, that is, $h(v) = \prod_{j=1}^n B(\alpha_j, v)$ where all $\alpha_j \in \mathbf{D}$, or $h(v) = \prod_{j=1}^n B(\beta_j, v)$ where all $\beta_j \in \mathbf{D}_+$. Then*

$$|h'(1)| = \|h\|_{\mathbf{T}} \max\left(\sum_{\alpha_j} \frac{\partial g_{\mathbf{D}}(1, \alpha_j)}{\partial \mathbf{n}_-}, \sum_{\beta_j} \frac{\partial g_{\mathbf{D}_+}(1, \beta_j)}{\partial \mathbf{n}_+} \right).$$

This proposition is contained in the Borwein-Erdélyi theorem as stated in [3] pp. 324-326.

First, we consider the case when

$$\Gamma \text{ is analytic and } Z \cap G_- \ne \emptyset\ , \tag{10.1}$$

where, as always, G_- is the interior domain determined by Γ.

Fix $z_0 \in \Gamma$, and let, as in Section 3.2, Φ_1 be the conformal map from the unit disk onto the interior domain G_- such that $\Phi_1(1) = z_0$, $|\Phi_1'(1)| = 1$. As has been discussed there, this Φ_1 can be extended to a disk $\{v \mid |v| < r_1\}$ with some $r_1 > 1$.

Let $\alpha_1, \dots, \alpha_n$ be n (not necessarily different) points from $\Phi_1^{-1}(Z \cap G_-)$, and let

$$h_n(v) := \prod_{j=1}^{n} B(\alpha_j, v),$$

for which $\|h_n\|_{\mathbf{T}} = 1$. Now we "transfer" h_n to G_- by considering $h_n\left(\Phi_1^{-1}(z)\right)$. If $f_{1,n}(z)$ is the sum of the principal parts of $h_n\left(\Phi_1^{-1}(z)\right)$ (with $f_{1,n}(\infty) = 0$), then

$$\varphi_e(z) := h_n\left(\Phi_1^{-1}(z)\right) - f_{1,n}(z)$$

is analytic in $G_1^+ := \{\Phi_1(v) \mid |v| < r_1\}$. Since h_n is at most 1 in absolute value outside the unit disk, it follows from Proposition 3.6 as in Section 4 that the absolute value of φ_ε is $\le C \log n$ on G_1^+. By Proposition 3.7 (applied to $K = \{\Phi_1(v) \bigm| |v| \le \sqrt{r_1}\}$ and to $\tau = \partial G_1^+$) there are polynomials $f_{2,\sqrt{n}}$ of degree at most $\sqrt{n}$ such that $f_{2,\sqrt{n}}(z_0) = \varphi_e(z_0)$, $f'_{2,\sqrt{n}}(z_0) = \varphi'_e(z_0)$ and

$$\|\varphi_e - f_{2,\sqrt{n}}\|_K \le C(\log n) q^{\sqrt{n}} \tag{10.2}$$

with some C and $q < 1$. Therefore, if we set

$$f_n(z) := f_{1,n}(z) + f_{2,\sqrt{n}},$$

then this is a rational function with poles in $Z \cap G_-$ of total degree n and with one pole at ∞ of order $\le \sqrt{n} = o(n)$. For it

$$|f_n'(z_0)| = |(h_n(\Phi_1^{-1}))'(z_0)| = |h_n'(1)|$$

since $|\Phi_1'(1)| = 1$. Furthermore $\|h_n\|_{\mathbf{T}} = 1$ (recall that $\mathbf{T}$ is the unit circle), so we obtain from (10.2)

$$\begin{aligned} \|f_n\|_\Gamma = \left\|f_{1,n} + f_{2,\sqrt{n}}\right\|_\Gamma &= \left\|f_{1,n} + \varphi_e + f_{2,\sqrt{n}} - \varphi_e\right\|_\Gamma \\ &= \left\|h_n(\Phi_1^{-1}) + f_{2,\sqrt{n}} - \varphi_e\right\|_\Gamma = 1 + O\left((\log n) q^{\sqrt{n}}\right) = 1 + o(1). \end{aligned}$$

We use Proposition 10.1 for h_n, hence

$$|f_n'(z_0)| = |h_n'(1)| = \|h_n\|_{\mathbf{T}} \sum_{\alpha_j} \frac{\partial g_{\mathbf{D}}(1, \alpha_j)}{\partial \mathbf{n}_-} \ge (1 - o(1)) \|f_n\|_\Gamma \sum_{\alpha_j} \frac{\partial g_{\mathbf{D}}(1, \alpha_j)}{\partial \mathbf{n}_-}.$$

Here, by Proposition 3.3,

$$\begin{aligned} \sum_{\alpha_j} \frac{\partial g_{\mathbf{D}}(1, \alpha_j)}{\partial \mathbf{n}_-} &= \sum_{\alpha_j} \frac{\partial g_{G_-}(z_0, \Phi_1(\alpha_j))}{\partial \mathbf{n}_-} \\ &= \max\left(\sum_{\alpha_j} \frac{\partial g_{G_-}(z_0, \Phi_1(\alpha_j))}{\partial \mathbf{n}_-}, \sqrt{n} \frac{\partial g_{G_+}(z_0, \infty)}{\partial \mathbf{n}_+}\right), \end{aligned}$$

where, in the last step, we used that the first term in the max is $\geq cn$ with some $c > 0$ (see (3.7)), so the last equality holds for large n.

Summarizing, we have proven that if Γ is an analytic Jordan curve, $Z \subset \overline{\mathbf{C}} \setminus \Gamma$ is a closed set, such that $Z \cap G_- \neq \emptyset$, then there exist rational functions $R_{n,-}$ with poles at any prescribed locations $a_{1,n}, \ldots, a_{n,n} \in Z \cap G_-$ and with a pole at ∞ of order $o(n)$ such that

$$\left|R'_{n,-}(z_0)\right| \geq (1 - o(1)) \left\|R_{n,-}\right\|_\Gamma \sum_{a_{j,n}} \frac{\partial g_{G_-}(z_0, a_{j,n})}{\partial \mathbf{n}_-}, \tag{10.3}$$

where $o(1)$ depends on Γ and Z only.

Similarly, if Γ is still an analytic Jordan curve and $Z \cap G_+ \neq \emptyset$, then the same assertion holds for some rational functions $R_{n,+}$ with prescribed poles at $a_{j,n} \in Z \cap G_+$ and with a pole of order $\leq \sqrt{n}$ at some given point ζ_0 inside Γ:

$$\left|R'_{n,+}(z_0)\right| \geq (1 - o(1)) \left\|R_{n,+}\right\|_\Gamma \sum_{a_{j,n}} \frac{\partial g_{G_+}(z_0, a_{j,n})}{\partial \mathbf{n}_+}. \tag{10.4}$$

This follows by applying a suitable inversion: fix $\zeta_0 \in G_-$ and apply the mapping $w = 1/(z - \zeta_0)$. We omit the details.

Now for analytic Γ Theorem 2.3 can be easily proven. For simplicity assume that the $a_{j,n}$ are different and finite (the following argument needs only simple modification if this is not the case). Suppose, for example, that for a given $n = 1, 2, \ldots$

$$\sum_{a_{j,n} \in Z \cap G_-} \frac{\partial g_{G_-}(z_0, a_{j,n})}{\partial \mathbf{n}_-} \geq \sum_{a_{j,n} \in Z \cap G_+} \frac{\partial g_{G_+}(z_0, a_{j,n})}{\partial \mathbf{n}_+}. \tag{10.5}$$

Consider the poles $a_{j,n}$ that are in G_-, and denote by $R_-(z)$ a rational function whose existence is established above for these poles (if the number of the $a_{j,n}$ that are in G_- is N, then in the previous notation this R_- is $R_{N,-}$, so the number of poles of R_- in G_- is N, and R_- also has a pole of order at most $\sqrt{N}$ at ∞). Next, for any given $\varepsilon > 0$ write

$$f_{n,+}(z) := \varepsilon_n \sum_{a_{j,n} \in Z \cap G_+} \frac{1}{z - a_{j,n}}$$

where $\varepsilon_n > 0$ is so small that $\|f_{n,+}\|_\Gamma \leq \varepsilon \|R_-\|_\Gamma$ and $\left|f'_{n,+}(z_0)\right| \leq \varepsilon \left|R'_-(z_0)\right|$. It is easy to see that then $R_n(z) := R_-(z) + f_{n,+}(z)$ has poles at the prescribed points $a_{1,n}, \ldots, a_{n,n}$ plus one additional pole of order $\leq \sqrt{n}$ at ∞. Furthermore, it satisfies

$$|R'_n(z_0)| \geq (1 - \varepsilon)^2 (1 - o(1)) \|R_n\|_\Gamma \sum_{a_{j,n} \in G_-} \frac{\partial g_{G_-}(z_0, a_{j,n})}{\partial \mathbf{n}_-},$$

and, by the assumption (10.5), the sum on the right is the same as the maximum in (2.3).

If (10.5) does not hold (i.e. the reverse inequality is true), then use the analogous R_+ ($= R_{n-N,+}$) and add to it a small multiple of the sum of $1/(z - a_{j,n})$ with $a_{j,n} \in Z \cap G_-$.

Since in these estimates $\varepsilon > 0$ is arbitrary, Theorem 2.3 follows for analytic Γ.

If Γ is not analytic, only C^2 smooth, then we can do the following. Suppose for example, that for an n (10.5) is true. For $\varepsilon > 0$ choose an analytic Jordan curve, say a lemniscate L, close to Γ such that $L \cap \Gamma = \{z_0\}$, $L \setminus \{z_0\}$ lies in the interior of Γ, and

$$(1-\varepsilon)\frac{\partial g_{G_-}(z_0,\beta)}{\partial \mathbf{n}_-} \le \frac{\partial g_{\overline{\mathbf{C}}\setminus L}(z_0,\beta)}{\partial \mathbf{n}_-} \tag{10.6}$$

for all $\beta \in G_- \cap Z$. (Here we used the shorthand notation $g_{\overline{\mathbf{C}}\setminus L}(z,a)$ for both $g_{\mathrm{Int}(L)}(z,a)$ when a is inside L and for $g_{\mathrm{Ext}(L)}(z,a)$ when a is outside L, where $\mathrm{Int}(L)$ and $\mathrm{Ext}(L)$ denote the interior and exterior domains to L.) The existence of L follows from the sharp form of Hilbert's lemniscate theorem in [16, Theorem 1.2] when $\beta = \infty$. For other β use fractional linear transformations to move the pole β to ∞, see the formula (10.9) below, as well as the reasoning there.

Now construct R_n for this L as before, and multiply it by a polynomial $Q = Q_{n^{7/8}}$ of degree at most $n^{7/8}$ such that $Q(1) = 1$, $\|Q\|_\Gamma \le 1$, and with some constants $c_0, C_0 > 0$

$$|Q(z)| \le C_0 \exp(-c_0 n^{7/8}|z - z_0|^{3/2}), \qquad z \in \Gamma.$$

Such a Q exists by [23, Theorem 4.1], and we have to consider R_nQ rather than R_n because the norm of R_n on Γ can be much larger than its norm on L, and Q brings that norm down, namely $\|R_nQ\|_\Gamma \le (1 + o(1))\|R_n\|_L$. Indeed, this is an easy consequence of (6.10) and Proposition 3.10 (both applied to L rather than Γ) and the properties of Q. Finally, since R_n proves Theorem 2.3 on L, relatively simple argument shows that R_nQ verifies it on Γ. The reader can easily fill in the details.

■

10.2 Sharpness of the Markov inequality

First we consider a C^2 Jordan curve γ and a point $z_0 \in \gamma$ on it. Let $\varepsilon > 0$. By the sharp form of the Hilbert lemniscate theorem [16, Theorem 1.2] there is a Jordan curve σ such that

- σ contains γ in its interior except for the point z_0, where the two curves touch each other,
- σ is a lemniscate, i.e. $\sigma = \{z \mid |T_N(z)| = 1\}$ for some polynomial T_N of degree N, and

- $$\frac{\partial g_{\overline{\mathbf{C}}\setminus\sigma}(z_0,\infty)}{\partial \mathbf{n}_+} \geq (1-\varepsilon)\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma}(z_0,\infty)}{\partial \mathbf{n}_+}, \tag{10.7}$$

  where the Green's functions $g_{\overline{\mathbf{C}}\setminus\gamma}(z_0,\infty)$ and $g_{\overline{\mathbf{C}}\setminus\sigma}(z_0,\infty)$ are taken with respect to the outer domains of γ and σ.

We may assume that $T_N(z_0) = 1$ and $T_N'(z_0) > 0$. The Green's function of the outer domain of σ is $\frac{1}{N}\log|T_N(z)|$, and its normal derivative is

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\sigma}(z_0,\infty)}{\partial \mathbf{n}_+} = \frac{1}{N}|T_N'(z_0)| = \frac{1}{N}T_N'(z_0).$$

Consider now, for all large n, the polynomials $S_n(z) = T_N(z)^{[n/N]}$, where $[n/N]$ denotes integral part. This is a polynomial of degree at most n, its supremum norm on σ is 1, and

$$S_n'(z_0) = \left[\frac{n}{N}\right] T_N(z_0)^{[n/N]-1}T_N'(z_0) = n\frac{\partial g_{\overline{\mathbf{C}}\setminus\sigma}(z_0,\infty)}{\partial \mathbf{n}_+} + O(1).$$

In a similar fashion,

$$\begin{aligned} S_n''(z_0) &= \left[\frac{n}{N}\right]\left(\left[\frac{n}{N}\right]-1\right) T_N(z_0)^{[n/N]-2}(T_N'(z_0))^2 + \left[\frac{n}{N}\right] T_N(z_0)^{[n/N]-1}T_N''(z_0) \\ &= n^2\left(\frac{\partial g_{\overline{\mathbf{C}}\setminus\sigma}(z_0,\infty)}{\partial \mathbf{n}_+}\right)^2 + O(n). \end{aligned}$$

Proceeding similarly, it follows that for any $j = 1, 2, \ldots$

$$S_n^{(j)}(z_0) = n^j\left(\frac{\partial g_{\overline{\mathbf{C}}\setminus\sigma}(z_0,\infty)}{\partial \mathbf{n}_+}\right)^j + O(n^{j-1}).$$

Thus, in view of (10.7), we may write

$$S_n^{(j)}(z_0) \geq (1-\varepsilon)^j n^j\left(\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma}(z_0,\infty)}{\partial \mathbf{n}_+}\right)^j + O(n^{j-1}), \tag{10.8}$$

where, and in what follows, we use the following convention: if A is a complex number and B is a positive number, then we write $A \geq B + O(n^s)$ if $A = C + O(n^s)$, where C is a real number with $C \geq B$. Note also that $\|S_n\|_\gamma \leq \|S_n\|_\sigma = 1$ by the maximum principle.

Next, we need an analogue of these for rational functions with pole at a point a that lies outside γ. Consider the fractional linear transformation $\varphi_a(z) = \xi/(z-a)$, where ξ is selected so that $|\xi| = 1$ and $\varphi_a'(z_0) > 0$. The image of γ

under this transformation is $\varphi_a(\gamma)$, and $g_{\overline{\mathbf{C}}\setminus\gamma}(z,a) = g_{\overline{\mathbf{C}}\setminus\varphi_a(\gamma)}(\varphi_a(z),\infty)$. This latter relation implies that

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma}(z_0,a)}{\partial \mathbf{n}_+(z_0)} = \frac{\partial g_{\overline{\mathbf{C}}\setminus\varphi_a(\gamma)}(\varphi_a(z_0),\infty)}{\partial \mathbf{n}_+(\varphi_a(z_0))}\varphi_a'(z_0). \tag{10.9}$$

Now let S_n be the polynomial constructed before, but this time for the curve $\varphi_a(\gamma)$ and for the point $\varphi_a(z_0)$, and set $S_{n,a}(z) = S_n(\varphi_a(z))$. This is a rational function with a pole of order at most n at a. Its norm on γ is at most 1, and, in view of (10.8) (applied to $\varphi_a(\gamma)$),

$$S_{n,a}'(z_0) = S_n'(\varphi_a(z_0))\varphi_a'(z_0) \geq (1-\varepsilon)n\frac{\partial g_{\overline{\mathbf{C}}\setminus\varphi_a(\gamma)}(\varphi_a(z_0),\infty)}{\partial \mathbf{n}_+(\varphi_a(z_0))}\varphi_a'(z_0) + O(1),$$

which can be written in the form

$$S_{n,a}'(z_0) \geq (1-\varepsilon)n\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma}(z_0,a)}{\partial \mathbf{n}_+} + O(1)$$

in view of (10.9). For the second derivative we have

$$S_{n,a}''(z_0) = S_n''(\varphi_a(z_0))(\varphi_a'(z_0))^2 + S_n'(\varphi_a(z_0))\varphi_a''(z_0),$$

hence

$$S_{n,a}''(z_0) \geq (1-\varepsilon)^2 n^2 \left(\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma}(z_0,a)}{\partial \mathbf{n}_+}\right)^2 + O(n),$$

in view of (10.8) (applied to $\varphi_a(\gamma)$ and to the point $\varphi_a(z_0)$) and (10.9). Proceeding similarly we obtain for all $j = 1, 2, \ldots$

$$S_{n,a}^{(j)}(z_0) \geq (1-\varepsilon)^j n^j \left(\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma}(z_0,a)}{\partial \mathbf{n}_+}\right)^j + O(n^{j-1}). \tag{10.10}$$

Now let there be given a fixed number of different poles $a_0, \ldots, a_m$ in the exterior of γ and associated orders $n_0, \ldots, n_m$, where $a_0 = \infty$ (if we do not want the point ∞ among the poles, just set $n_0 = 0$). For the total degree $n = n_0 + \cdots + n_m$ consider the rational function

$$U_n(z) = \prod_{i=0}^{m} S_{n_i,a_i}(z),$$

where we set $S_{n_0,a_0} = S_{n_0}$, with the polynomial S_{n_0} constructed in the first part of the proof. This is a rational function with poles at the a_i's and the order of a_i is at most n_i. Since

$$U_n^{(k)}(z_0) = \sum_{j_0+\cdots+j_m=k} \frac{k!}{j_0!\cdots j_m!} S_{n_0,a_0}^{(j_0)}(z_0)\cdots S_{n_m,a_m}^{(j_m)}(z_0),$$

we obtain from (10.10) that

$$\begin{aligned}
U_n^{(k)}(z_0) &\geq \sum_{j_0+\cdots j_m=k} \frac{k!}{j_0!\cdots j_m!} \prod_{i=0}^{m} \left((1-\varepsilon)^{j_i} n_i^{j_i} \left(\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma}(z_0,a_i)}{\partial \mathbf{n}_+} \right)^{j_i} + O(n_i^{j_i-1}) \right) \\
&= \sum_{j_0+\cdots j_m=k} \frac{k!}{j_0!\cdots j_m!} \prod_{i=0}^{m} \left((1-\varepsilon)^{j_i} n_i^{j_i} \left(\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma}(z_0,a_i)}{\partial \mathbf{n}_+} \right)^{j_i} \right) + O(n^{k-1}),
\end{aligned}$$

so by the multinomial theorem (see e.g. [12, Theorem 1.3.1])

$$U_n^{(k)}(z_0) \geq (1-\varepsilon)^k \left(\sum_{i=0}^{m} n_i \frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma}(z_0,a_i)}{\partial \mathbf{n}_+} \right)^k + O(n^{k-1}).$$

Hence,

$$|U_n^{(k)}(z_0)| \geq (1-\varepsilon)^k (1-o(1)) \left(\sum_{i=0}^{m} n_i \frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma}(z_0,a_i)}{\partial \mathbf{n}_+} \right)^k \tag{10.11}$$

in view of (3.8).

After these preparations we can prove the last statement in Theorem 2.8. Let Γ be a C^2 smooth Jordan arc and $a_0,\ldots,a_m$ be finitely many fixed poles outside Γ with associated orders $n_0,\ldots,n_m$. We agree that $a_0=\infty$, and if we do not want the point ∞ among the poles, just set $n_0=0$. We may assume that the endpoint A of Γ is at the origin, and consider, as before, the curve $\Gamma^* = \{z \mid z^2 \in \Gamma\}$. We also consider the poles $\pm\sqrt{a_i}$, $i=0,\ldots,m$, with associated orders n_i with the agreement that if $n_0 \neq 0$, i.e. the point ∞ is among our poles, then $\pm\sqrt{\infty}=\infty$.

It is easy to see that there is a C^2 Jordan curve γ such that

- γ contains Γ^* in its interior except for the point 0, where γ and Γ^* touch each other,
- all a_i are outside γ,
- $$\frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma}(0,a_i)}{\partial \mathbf{n}_+} \geq (1-\varepsilon) \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma^*}(0,a_i)}{\partial \mathbf{n}_+} \quad \text{for all } i. \tag{10.12}$$

Indeed, all we need to do is to select γ sufficiently close to Γ^* and to have at 0 curvature close to the curvature of Γ^*, see e.g. [16]. Now apply (10.11) to this γ, to z_0 and to the poles $\pm\sqrt{a_i}$ with the associated orders n_i, but for the $2k$-th derivative. We get a rational function U_{2n}, $n=n_0+\cdots+n_m$, with poles at $\pm\sqrt{a_i}$ of order at most n_i such that $\|U_{2n}\|_\gamma \leq 1$ and

$$|(U_{2n}(z))^{(2k)}(0)| \geq (1-\varepsilon)^{2k}(1-o(1)) \left(\sum_{i=0}^{m} n_i \left\{ \frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma}(0,\sqrt{a_i})}{\partial \mathbf{n}_+} + \frac{\partial g_{\overline{\mathbf{C}}\setminus\gamma}(0,-\sqrt{a_i})}{\partial \mathbf{n}_+} \right\} \right)^{2k},$$

which yields, in view of (10.12),

$$|(U_{2n}(z))^{(2k)}(0)| \geq (1-\varepsilon)^{4k}(1-o(1)) \left(\sum_{i=0}^{m} n_i \left\{ \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma^*}(0, \sqrt{a_i})}{\partial \mathbf{n}_+} + \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma^*}(0, -\sqrt{a_i})}{\partial \mathbf{n}_+} \right\} \right)^{2k}. \tag{10.13}$$

Note also that, by the maximum principle, we have $\|U_{2n}\|_{\Gamma^*} \leq \|U_{2n}\|_\gamma \leq 1$ because all the poles of U_{2n} lie outside γ.

By the symmetry of Γ^* and of the system $\{\pm\sqrt{a_i}\}$ onto the origin then $U_n(-z)$ also has this property, furthermore $(U_{2n}(-z))^{(2k)}(0) = (U_{2n}(z))^{(2k)}(0)$, so if we set $\mathcal{R}_{2n}(z) = \frac{1}{2}(U_{2n}(z)+U_{2n}(-z))$, then $\mathcal{R}_{2n}$ is an even rational function for which (10.13) is true if we replace in it $U_{2n}(z)$ by $\mathcal{R}_{2n}(z)$. But then there is a rational function R_n such that $\mathcal{R}_{2n}(z) = R_n(z^2)$, and for this R_n we have that $\|R_n\|_\Gamma = \|\mathcal{R}_{2n}\|_{\Gamma^*} \leq 1$, and (see (10.13))

$$(R_n(z^2))^{(2k)}(0) \geq (1-\varepsilon)^{4k}(1-o(1))2^{2k} \left(\sum_{i=0}^{m} n_i \Omega_{a_i}(A) \right)^{2k}$$

where we used the equality of the two quantities in (9.2) and (9.6). Note also that this R_n has poles at $a_0, \ldots, a_m$ of orders at most $n_0, \ldots, n_m$. Now the argument used in the proof of Theorem 2.8 via the Faà di Bruno's formula shows that the preceding inequality is the same as

$$|R_n^{(k)}(0)| \geq (1-\varepsilon)^{4k}(1-o(1)) \frac{2^k}{(2k-1)!!} \left(\sum_{i=0}^{m} n_i \Omega_{a_i}(A) \right)^{2k}.$$

A similar construction can be done for the other endpoint B of Γ, and by taking the larger of the two expressions in these lower estimates we finally conclude the last statement in Theorem 2.8 regarding the sharpness of (2.13).

∎

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