Higher Markov and Bernstein inequalities and fast decreasing polynomials with prescribed zeros

Sergei Kalmykov and Béla Nagy

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Abstract

Higher order Bernstein- and Markov-type inequalities are established for trigonometric polynomials on compact subsets of the real line and algebraic polynomials on compact subsets of the unit circle. In the case of Markov-type inequalities we assume that the compact set satisfies an interval condition.

Keywords: trigonometric polynomials, algebraic polynomials, Bernstein-type inequalities, equilibrium measure, Green’s function, fast decreasing polynomials.

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1 Introduction

Two of the most classical polynomial inequalities are the Bernstein inequality (see [2], p. 233 Theorem 5.1.7 or [14], p. 532, Theorem 1.2.5)

\[ |P'_n(x)| \leq \frac{n}{\sqrt{1-x^2}} \|P_n\|_{[-1,1]}, \quad x \in (-1,1), \]

and the Markov inequality (see [2], p. 233 Theorem 5.1.8 or [14], p. 529 Theorem 1.2.1)

\[ \|P'_n\|_{[-1,1]} \leq n^2 \|P_n\|_{[-1,1]}, \]

where \(P_n\) is an algebraic polynomial of degree of at most \(n\), and \(\| \cdot \|_X\) denotes the sup-norm over the set \(X\). For a trigonometric polynomial \(T_n\) of the degree at most \(n\) the following Bernstein-type inequality holds (established by M. Riesz, see [14], p. 532 Theorem 1.2.4 or [2], p. 232 Theorem 5.1.4)

\[ \|T'_n\|_{[0,2\pi]} \leq n \|T_n\|_{[0,2\pi]}, \]

There is also an analogue of this inequality for trigonometric polynomials on an interval less than the period see [2] p. 243. In 2001, Totik developed the method of polynomial inverse images to prove an asymptotically sharp Bernstein- and Markov-type inequalities for algebraic polynomials on several intervals [25], and in [28] asymptotically sharp inequalities were also obtained.
for trigonometric polynomials on several intervals and for algebraic polynomials on several circular arcs on the complex plane. The case of one circular arc was considered earlier in [16]. In recently published paper [7] algebraic polynomials on sets satisfying (2) were considered, for trigonometric polynomials, see [6]. The next step in generalization of these result was done in [23], where asymptotic higher order Markov-type inequalities for algebraic polynomials on compact sets satisfying (2) were established.

The purpose of the present paper is to extend these results to trigonometric polynomials and to algebraic polynomials on subsets of the unit circle and to present a new type of fast decreasing polynomials. Briefly, the approach of Totik-Zhou [23] was to establish the Markov-type inequality for T-sets, then for general sets and use Faà di Bruno’s formula and Remez inequality near interior critical points. The difference here is that we developed fast decreasing polynomials with prescribed zeros to deal with interior critical points. Moreover, we also establish Bernstein-type inequality.

Sharp higher order Markov-type inequality is established for sets satisfying the interval condition (2). At interior points sharp Bernstein-type inequality is also derived which involves much slower growth order (\(O(n^{2k})\) at endpoints vs. \(O(n^k)\) at interior points where \(k\)-th derivatives are considered).

The structure of the paper is the following. First, notation is introduced, and some known, basic results about T-sets are mentioned. Then the important density results (for T-sets and regular sets) are recalled. New results are in Section 3. A construction of fast decreasing polynomials with prescribed zeros can also be found here. A preliminary, "rough" Markov- and Bernstein-type inequalities are needed for special sets. Then asymptotically sharp Markov-type inequality is formulated for higher derivatives of trigonometric polynomials and for algebraic polynomials on subsets of the unit circle. Finally, asymptotically sharp Bernstein-type inequalities are established in the trigonometric case as well as in the algebraic case.

2 Notation, background

We denote by \(\mathbb{R}\) the real line, by \(\mathbb{C}\) the complex plane, by \(\overline{\mathbb{C}}\) the extended complex plane, and by \(\mathbb{T}\) the unit circle and by \(\mathbb{N}\) the nonnegative integers.

We use Faà di Bruno’s formula (or Arbogast’s formula; see [9], p. 17 or [21], pp. 35-37 or [23]): if \(f\) and \(g\) are \(k\) times differentiable functions, then

\[
\frac{d^k}{dx^k} f(g(x)) = \sum \frac{k!}{m_1!m_2! \cdots m_k!} f^{(m_1+m_2+\ldots+m_k)}(g(x)) \prod_{j=1}^{k} \left( \frac{g^{(j)}(x)}{j!} \right)^{m_j}
\]

(1)

where the summation is for all nonnegative integers \(m_1, m_2, \ldots, m_k\) such that

\[1m_1 + 2m_2 + \ldots + km_k = k.\]

Let \(E \subset [-\pi, \pi]\) be a set which is closed in \([-\pi, \pi]\). Since we do not consider \(E = [-\pi, \pi]\) (it is classical), we may assume that \(E \subset (-\pi, \pi)\). We consider the
corresponding set on the unit circle

\[ E_T := \{ \exp(it) : t \in E \}. \]

We use the interval condition: a compact set \( E \subset (-\pi, \pi) \) satisfies the interval condition at \( a \in E \) if there is a \( \rho > 0 \) such that

\[ [a - 2\rho, a] \subset E \quad \text{and} \quad (a, a + 2\rho) \cap E = \emptyset. \quad (2) \]

We use potential theory, for a background, we refer to [20] or [22]. For a compact set \( K \subset \mathbb{C} \), its capacity is denoted by \( \text{cap}(K) \). If \( \text{cap}(K) > 0 \), then the equilibrium measure is denoted by \( \nu_K \). It is known that if \( K \subset \mathbb{R} \) is a compact set \( \nu_K \) is absolutely continuous with respect to Lebesgue measure at interior points of \( K \) and its density is denoted by \( \omega_K \).

\[ \Omega(\cdot) \]

\[ \Omega(E, a) := \lim_{t \to a} \sqrt{|t - a|} \omega_E(t), \]

\[ \Omega(K, e^{ia}) := \lim_{t \to a} \sqrt{|e^{it} - e^{ia}|} \omega_K(e^{it}). \]

It is worth noting that \( \Omega(\cdot) \) is monotone with respect to the set, that is, if \( E_1 \subset E_2 \subset (-\pi, \pi) \), and both satisfy the interval condition at \( a \), then \( \Omega(E_2, a) \leq \Omega(E_1, a) \). Similar assertion holds for the unit circle.

In the finitely many arcs case, there is a very useful representation of the density of the equilibrium measure (see [19], Lemma 4.1 and also formula (5.11)): let \( K = \cup_{j=1}^{m} \{ \exp(it) : a_{2j-1} \leq t \leq a_{2j} \} \) where \( -\pi < a_1 < a_2 < \ldots < a_{2m-1} < a_{2m} < \pi \) and put \( a_{2m+1} := 2\pi + a_1 \). Then there exist \( \tau_j \in (a_{2j}, a_{2j+1}) \), \( j = 1, \ldots, m \) such that

\[ \int_{a_{2j}}^{a_{2j+1}} \frac{\prod_{j=1}^{m} (e^{it} - e^{i\tau_j})}{\sqrt{\prod_{j=1}^{m} (e^{it} - e^{ia_{2j-1}})(e^{it} - e^{ia_{2j}})}} dt = 0 \quad (3) \]

where, to be definite, the branch of the square root is chosen so that \( \sqrt{z} \to \infty \) as \( z \in \mathbb{R}, z \to +\infty \). Actually it should hold that

\[ \frac{(-1)^m}{i} \prod_{j} e^{i\tau_j} = \sqrt{\prod_{j} e^{i(a_{2j-1} + a_{2j})}} \]

but actually the other branch would be just as fine, since the right hand side in (3) is 0. Then

\[ \omega(K, e^{it}) = \frac{1}{2\pi} \prod_{j=1}^{m} \frac{|e^{it} - e^{i\tau_j}|}{\sqrt{\prod_{j=1}^{m} |e^{it} - e^{ia_{2j-1}}||e^{it} - e^{ia_{2j}}|}}, \quad t \in \text{Int } K. \]
see [19], formula (5.11). In this case,

$$\Omega(K, e^{ia_k}) = \frac{1}{2\pi} \frac{\prod_{j=1}^{m} |e^{ia_k} - e^{i\tau_j}|}{\prod_{j=1,...,2m,j\neq k} |e^{ia_k} - e^{ia_l}|}.$$ 

### 2.1 Density results

We use special sets on \((-\pi, \pi)\). A set \(E \subset (-\pi, \pi)\) is called T-set, if

\[
E = \{ t \in (-\pi, \pi) : |U_N(t)| \leq 1 \}
\]

for some (real) trigonometric polynomial \(U_N\) with degree \(N\) which attains +1 and −1 \(2N\)-times. For a background on T-sets, we refer to Section 3 in [28].

We define

\[
M(E, a_j) = M_{a_j} := \frac{\prod_{l=1}^{m} |e^{ia_j} - e^{i\tau_l}|^2}{|e^{ia_j} - e^{ia_l}|}
\]

and obviously,

\[
M(E, a_j) = 4\pi^2 \Omega^2(E_T, e^{ia_j}).
\]

Now we recall some monotonicity and continuity results regarding \(\Omega(E, a)\) and \(M(E, a)\).

For any \(\varepsilon > 0\), by Lemma 3.4 from [28] (see p. 3001) we can choose an admissible polynomial \(U_N\) such that the inverse image set \(E' = (U_N^{-1}[-1, 1]) \cap [-\pi, \pi] = \bigcup_{l=1}^{m} [a_{j_l-1}, a_{j_l}]\) consists of \(m\) intervals and it lies close to \(E\), that is \(|a_j' - a_j| < \varepsilon\) for all \(j = 1, \ldots, 2m\) and \(E' \subset E\). Also we may assume that \(a \in E'\). Again \(j_0\) is such that \(a \in [a_{j_0-1}, a_{j_0}]\) and actually \(a = a_{j_0}'\). For numbers \(\tau_i\) in \([3]\), it is clear that they are \(C^1\)-functions of the endpoints \(a_j\). Then with \(M_a' := M(E', a)\), we have \(\lim_{\varepsilon \to 0} M_a' = M_a\). By the monotonicity of \(\Omega(\cdot, \cdot)\) in the first variable, we immediately have that \(M_a \leq M_a'\).

In other words, for any \(\varepsilon > 0\), there exists a T-set \(E' \subset E, a \in E'\) such that \(\Omega^2(E_T, e^{ia}) \leq (1 + \varepsilon)\Omega^2(E_T, e^{ia})\).

Consider an arbitrary compact set \(E \subset (-\pi, \pi)\) satisfying the interval condition \([2]\), and assume that \(E\) is not a union of finitely many intervals. The set \([-\pi, \pi] \setminus E\) consists of finitely or countably many intervals open in \([-\pi, \pi]\):

\([-\pi, \pi] \setminus E = \bigcup_{j=0}^{\infty} I_j\]

To be definite, we assume that \(I_0\) contains \((a, a + 2\rho)\). Further, for \(m \geq 0\) we consider the set

\[
E_m^+ = [-\pi, \pi] \setminus \left( \bigcup_{j=0}^{m} I_j \right) = \bigcup_{j=1}^{m} [a_{j,m'}, b_{j,m'}],
\]

\[
a_{1,m'} \leq b_{1,m'} < a_{2,m'} \leq b_{2,m'} < \cdots < a_{m',m'} \leq b_{m',m'} = b_0,m'
\]
where \( m' = m + 1 \) (note here, by our assumption \( E \subset (-\pi, \pi) \)).

Obviously, \( E_{m}' \) contains \( E \) and satisfies the interval condition (2). If \( a_{j, m'} = b_{j, m'} \) for some \( j \), then we replace this degenerated interval by the interval

\[
[a_{j, m'} - \lambda_m, a_{j, m'} + \lambda_m] \cap [-\pi, \pi],
\]

where \( \lambda_m < 1/m \) is chosen to be so small that the interval condition (2) is still satisfied. For the set obtained this way we preserve the notation \( E_{m}' \).

We also use the famous result of Ancona (see \( \text{[1]} \)). If \( K \subset \mathbb{T} \) is any compact set, \( \text{cap}(K) > 0 \), then for any \( \varepsilon > 0 \) there exists \( K_1 \subset K \) compact set which is regular for the Dirichlet problem and \( \text{cap}(K) \leq \text{cap}(K_1) + \varepsilon \). Furthermore, it is easy to see that if \( K \) satisfies the interval condition (2), then \( K_1 \) can be chosen such that it satisfies (2) too. Let \( E_{m}' \) be the set coming from Ancona’s theorem applied to \( E_T \) with \( \varepsilon = 1/m \) and also satisfying the interval condition (2).

**Lemma 1.** For the two sets \( E_{m}^{+} \) and \( E_{m}^{-} \) introduced above, we have \( \Omega((E_{m}^{+})_{\mathbb{T}}, e^{ia}) \rightarrow \Omega(E_{T}, e^{ia}) \) holds true as \( m \rightarrow \infty \).

For a proof, see e.g. \( \text{[7]} \), p. 1295, Proposition 2.3.

### 3 New results

We need fast decreasing polynomials with prescribed zeros and rough Markov- and Bernstein-type inequalities.

#### 3.1 Fast decreasing trigonometric and algebraic polynomials with prescribed zeros

Special fast decreasing polynomials with prescribed zeros are constructed in this subsection. First, their existence are established on the real line, then in the trigonometric case.

We tried to find this type of fast decreasing polynomials in the existing literature (e.g. in \( \text{[12]}, \text{[4]}, \text{[24]}, \text{[26]}, \text{[27]}, \text{[29]}, \text{[10]} \) and Lemma 4.5 on p. 3012 in \( \text{[28]} \)), but we did not find the following two results. Further, possible applications may include estimates for Christoffel functions, etc.

**Theorem 2.** Let \( a_0 < a_1 < \ldots < a_{l_0} < a' < a < x_0 < b < b' < a_{l_0+1} < \ldots < a_{l+1} \) be fixed and \( k_0, k_1, \ldots, k_l \) be positive integers. Put \( Z(x) := \prod_{j=1}^{l}(x-a_j)^{k_j} \). Then there exists \( \delta_1 > 0 \) such that for all large \( m \) there exists
a polynomial $Q(x)$ with degree at most $m$ such that

$$Q(x_0) = 1,$$

$$Q^{(j)}(x_0) = 0, \quad j = 1, \ldots, k_0,$$

$$|Q(x)| < 1 \text{ if } x \in [a_0, a_{l+1}], x \neq x_0,$$

$$|Q(x) - 1| \leq \exp(-\delta_1 m) \text{ for } x \in [a, b],$$

$$|Q(x)| \leq \min(1, |Z(x)|) \exp(-\delta_1 m) \text{ for } x \in [a_0, a'] \cup [b', a_{l+1}],$$

$$Q^{(k)}(a) = 0, \quad j = 1, \ldots, l, k = 0, 1, \ldots, k_j,$$

$$Q(x) \geq 0 \text{ for } x \in [a_0, a_{l+1}].$$

**Proof.** In this proof several new pieces of notation are introduced which are used here only and constants are not redefined from line to line in this proof just for sake of convenience.

Consider $S$, which will be a polynomial satisfying all but one properties, in the form

$$S(x) = C_1 \int_{a_1}^{x} Z_1(t) \, P_1(t) \, R(t)(t - x_0)^{k_0} \, dt$$

where

$$Z_1(t) := \prod_{j=1}^{l} (t - a_j)^{k_j}, \quad R(\tau; t) = R(t) := \prod_{j=1}^{l-1} (t - \tau_j)$$

$$P_0(t) = P_0(\delta, \mu; t) := \left(1 - \frac{(x - \delta)}{c_2}\right)^{2\mu}$$

$$P_1(\alpha, \beta, \lambda, \mu; t) = (1 - \lambda)P_0(\alpha, \mu; t) + \lambda P_0(\beta, \mu; t)$$

and where $k_0 = k_0'$ if $k_0$ is odd and $k_0 = k_0 + 1$ if $k_0$ is even, and for $j = 1, \ldots, l$, $k_j = k_j'$ if $k_j$ is even and $k_j' = k_j + 1$ if $k_j$ is odd, and $\tau_j \in [a_j, a_{j+1}], j = 1, \ldots, l - 1, j \neq l_0$, and $a' < \alpha < a < b < \beta < b'$, $\alpha := (a + a')/2, \beta := (b + b')/2$ and $\mu$ is large positive integer and $c_2 := a_{l+1} - a_0, \lambda \in [0, 1]$. If some of the parameters are fixed or unimportant in the current consideration, then we leave them out, e.g. $P_0(t) = P_0(\delta, \mu; t)$ and $P_1(t) = P_1(\mu; t) = P_1(\lambda, \mu; t) = P_1(\alpha, \beta, \lambda, \mu; t)$.

The key observation is that if $S(a_j) = 0$ for some $j$, then we immediately have that $S(k_0)(a_j) = 0, k = 0, 1, 2, \ldots, k_j$.

Some obvious properties immediately follow from the definitions: $Z_1(t) \geq 0$ (this is why we increased the "multiplicities"), $P_0(t), P_1(t) \geq 0$ too, $\max_{a_0 \leq t \leq a_{l+1}} P_1(t) \geq 1/2$. Furthermore, the degree of $R$ is $l - 2$ and $R$ has the same sign over $(a', b')$. For simplicity, denote $\mathcal{I}_1 := (\tau_1, \ldots, \tau_{l_0-1}), \mathcal{I}_2 := (\tau_{l_0+1}, \ldots, \tau_l)$ and (slightly abusing the notation) $\mathcal{I} := (\tau_1, \tau_2) = (\tau_1, \tau_{l_0-1}, \tau_{l_0+1}, \ldots, \tau_l)$ and $(\mathcal{I}_1, \mathcal{I}_2) := (\tau_1, \ldots, \tau_{l_0-1}, \lambda, \tau_{l_0+1}, \ldots, \tau_l)$. Finally, the degree of $S$ is $k_1' + \ldots + k_l' + 2\mu + l - 2 + k_0 + 1 = 2\mu + \text{const.}$
Poincaré-Miranda theorem (see e.g. [11], p. 547 or [18], pp. 152-153) helps to find a solution so that \( S \) vanishes at all prescribed \( a_j \)'s. In detail, put \( \mathcal{R} := [a_1, a_2] \times \cdots \times [a_{n_0-1}, a_{n_0}] \times [0, 1] \times [a_{n_0+1}, a_{n_0+2}] \times \cdots \times [a_{l-1}, a_l] \) and for \( j = 1, \ldots, l \) let \( f_j : \mathcal{R} \to \mathbb{R} \),

\[
f_j(\tau_1, \lambda, \tau_2) := \int_{a_j}^{a_{j+1}} Z_1(t) P_1(\lambda, \mu; t) R(\tau; t)(t - x_0)^{k_0} dt.
\]

Now we verify the signs of these functions on opposite sides of \( \mathcal{R} \): if \( j = 1, \ldots, l \), \( j \neq l_0 \), then \( A_j := \{(\tau_1, \cdots, \tau_{n_0-1}, \lambda, \tau_{n_0+1}, \cdots, \tau_l) \in \mathcal{R} : \tau_j = a_j \} \) and \( B_j := \{(\tau_1, \cdots, \tau_{n_0-1}, \lambda, \tau_{n_0+1}, \cdots, \tau_l) \in \mathcal{R} : \tau_j = a_{j+1} \} \) are the opposite sides. We have to treat the case \( j < l_0 \) and the case \( j > l_0 \) separately. If \( (\tau_1, \lambda, \tau_2) \in A_j \), then \( R(t) \) has the same sign all over \((a_j, a_{j+1})\) and \( f_j(\tau_1, \lambda, \tau_2) = \text{sign} R(t)(t - x_0)^{k_0} = (-1)^{l-1-j+k_0} = (-1)^{l-j} \) if \( j < l_0 \) and \( f_j(\tau_1, \lambda, \tau_2) = \text{sign} R(t) = (-1)^{l-1-j} \) if \( j > l_0 \). On the other side, if \( (\tau_1, \lambda, \tau_2) \in B_j \), then this means that we move \( \tau_j \) from \( a_j \) to \( a_{j+1} \) hence the sign of \( R(t) \) changes. That is, the sign of \( R(t) \) is the same as that of \( f_j(\tau_1, \lambda, \tau_2) \), hence if \( j < l_0 \), then \( \text{sign} f_j(\tau_1, \lambda, \tau_2) = \text{sign} R(t)(t - x_0)^{k_0} = (-1)^{l-j+1} \) and if \( j > l_0 \), then \( \text{sign} f_j(\tau_1, \lambda, \tau_2) = (-1)^{l-j} \), which shows the sign change in both cases (when \( j = 1, \ldots, l_0 - 1 \) and when \( j = l_0 + 1, \ldots, l \)).

As regards \( j = l_0 \), we estimate \( Z_1(t) \) and \( R(t) \) first. Let \( \rho_1 := 1/4 \min(a - a', x_0 - a, b - x_0, b' - b) > 0 \). Considering \( Z_1(t) \), it is easy to see that there exists \( C_3 > 0 \) such that for all \( t \in [\alpha_r - \rho_1, \alpha_r + \rho_1] \cup [\beta - \rho_1, \beta + \rho_1] \) we have \( 1/C_3 \leq Z_1(t) \leq C_3 \). The family of possible polynomials \( R(\tau; t) \) also has this property: there exists \( C_4 > 0 \) such that for any \( (\tau_1, \lambda, \tau_2) \in \mathcal{R} \), and for any \( t \in [\alpha - \rho_1, \alpha + \rho_1] \cup [\beta - \rho_1, \beta + \rho_1] \) we have \( 1/C_4 \leq |R(\tau; t)(t - x_0)^{k_0}| \leq C_4 \). Now we need Nikolskii inequality to give a lower estimate for the integral of \( P_0 \) near \( \alpha \) and \( \beta \). Using that \( ||P_0(\alpha, \mu, \cdot)||_{[\alpha - \rho_1, \alpha + \rho_1]} = P_0(\alpha) = 1 \) and \( \deg(P_0) = 2\mu \), Nikolskii inequality (see e.g. [12], p. 498, Theorem 3.1.4.) yields that there exists \( C_5 > 0 \) independent of \( \mu \) and \( P_0 \) such that

\[
\int_{\alpha - \rho_1}^{\alpha + \rho_1} P_0(\alpha, \mu; t) dt = \int_{\alpha - \rho_1}^{\alpha + \rho_1} |P_0(\alpha, \mu; t)| dt \geq C_5 \frac{1}{\mu^2}
\]

with some \( C_5 > 0 \) depending on \( \rho_1 \) only and we can easily obtain

\[
\int_{\alpha - \rho_1}^{\alpha + \rho_1} P_0(\alpha, \mu; t) Z_1(t) |R(\tau; t)(t - x_0)^{k_0}| dt \geq \frac{C_5}{C_3 C_4} \frac{1}{\mu^2}
\]

(14)

as well. Moreover, for any \( \lambda \in [0, 1], \max_{[\alpha - \rho_1, \alpha + \rho_1]} P_1(\cdot) \geq 1 - \lambda \), hence applying Nikolskii inequality (see e.g. [12], p. 498, Theorem 3.1.4.) on these intervals,

\[
\int_{\alpha - \rho_1}^{\alpha + \rho_1} P_1(\lambda, \mu; t) Z_1(t) |R(\tau; t)(t - x_0)^{k_0}| dt \geq \frac{C_5}{C_3 C_4} \frac{1 - \lambda}{\mu^2}
\]

and similarly for \([\beta - \rho_1, \beta + \rho_1]\).
We need an upper estimate too. If \( t \in [a_0, a_{l+1}] \), \( |t - \alpha| \geq \rho_1 \), then with 
\[ \rho_2 := 1 - \left( \frac{\rho_1}{\rho_2} \right)^2 < 1 \] we can write

\[
P_0(\alpha, \mu; t) \leq \rho_2^2
\]

and if \( t \in H := [a_0, \alpha - \rho_1] \cup [\alpha + \rho_1, \beta - \rho_1] \cup [\beta + \rho_1, a_{l+1}] \) then

\[
P_0(\alpha, \mu; t)Z_1(t) \left| R(t)(t - x_0)^{k_0} \right|, P_0(\beta, \mu; t)Z_1(t) \left| R(t)(t - x_0)^{k_0} \right| \leq C_3 C_4 \rho_2^2
\]

and

\[
P_0(\alpha, \mu; t)Z_1(t) \left| R(t)(t - x_0)^{k_0} \right| \leq C_3 C_4 \rho_2^2, \ |t - \beta| \leq \rho_1, \tag{16}
\]

\[
P_0(\beta, \mu; t)Z_1(t) \left| R(t)(t - x_0)^{k_0} \right| \leq C_3 C_4 \rho_2^2, \ |t - \alpha| \leq \rho_1. \tag{17}
\]

Now we can investigate \( f_{l_0}(.) \) on \( A_{l_0} := \{(\tau_1, \ldots, \tau_{l_0-1}, \lambda, \tau_{l_0+1}, \ldots, \tau_l) \in R : \lambda = 0\} \); by \([14]\) we can write

\[
\left| \int_{a_{l_0}}^{x_0} P_0(\alpha, \mu; t)Z_1(t)R(\tau; t)(t - x_0)^{k_0} dt \right| \geq \int_{\alpha - \rho_0}^{\alpha + \rho_0} P_0(\alpha, \mu; t)Z_1(t) \left| R(\tau; t)(t - x_0)^{k_0} \right| dt \geq \frac{C_5}{C_3 C_4} \frac{1}{\mu^2}
\]

and by \([15]\), we can write

\[
\left| \int_{x_0}^{a_{l_0} + 1} P_0(\alpha, \mu; t)Z_1(t)R(\tau; t)(t - x_0)^{k_0} dt \right| \leq c_2 C_3 C_4 \rho_2^2.
\]

These last two displayed estimates show that \( f_{l_0}(.) \) on \( A_{l_0} \) has the same sign as \( R(t)(t - x_0)^{k_0} \) on \((a_{l_0}, x_0)\) (that is, \((-1)^{l-l_0-1+k_0} = (-1)^{l-l_0}\) if \( \mu \) is large \((\mu \geq \mu_1)\). Similarly, by replacing \( \alpha \) with \( \beta \), we can say that \( f_{l_0}(.) \) on \( B_{l_0} := \{(\tau_1, \ldots, \tau_{l_0-1}, \lambda, \tau_{l_0+1}, \ldots, \tau_l) \in R : \lambda = 1\} \) has the same sign as \( R(t)(t - x_0)^{k_0} \) on \((x_0, a_{l_0+1})\) (that is, \((-1)^{l-l_0+1}\) again if \( \mu \) is large \((\mu \geq \mu_2)\). These two observations show that on the opposite sides \( A_{l_0} \) and \( B_{l_0} \), \( f_{l_0}(.) \) has different signs (since \( k_0 \) is odd). Obviously, all \( f_j(.) \) functions are continuous.

Now the conditions of Poincaré-Miranda theorem are satisfied, hence there exists \((\tau_1, \lambda, \tau_2) \in R\) such that \( f_j(\tau_1, \lambda, \tau_2) = 0 \) for all \( j = 1, \ldots, l \). Fix these values and denote them by the same letters in the rest of this proof.

Finally, in \([13]\), we choose \( C_1 \in \mathbb{R} \) so that \( S(x_0) = 1 \), where actually we can write

\[
\frac{1}{C_1} = \int_{a_{l_0}}^{x_0} P_1(\lambda, \mu; t)Z_1(t)R(\tau; t)(t - x_0)^{k_0} dt
\]

and by knowing the sign of \( R(\tau; .) \) over \((a_{l_0}, x_0)\), sign \( C_1 = (-1)^{l-l_0+k_0} = (-1)^{l-l_0} \) and by \([14]\), \( |C_1| = O(\mu^2) \).
So $S$ is uniquely determined and it has the following properties. $S(a_j) = 0$ for all $j = 1, \ldots, l$, hence by the key observation, (11) holds. By the normalization (5) is true. (6) is also true, because of (13). For simplicity, put

$$S_1(t) := C_1 Z_1(t) P_1(t) R(t)(t - x_0)^{k_0}.$$ 

To see (7), (8), (10), and the first half of (9) (with 1 in place of $\min(1, |Z(x)|)$) first note that (15) implies that

$$|Z_1(t) P_1(\mu; t) R(t)(t - x_0)^{k_0}| \leq C_3 C_4 \rho_2^k$$

when $t \in H = [a_0, \alpha - \rho_1] \cup [\alpha + \rho_1, \beta - \rho_1] \cup [\beta + \rho_1, a_{l+1}].$ Moreover, let us remark that

$$|P_1(\mu; t)| \leq \rho_2^k$$

for $t \in H$. Let us choose $\delta_1 > 0$ such that $0 < \delta_1 < -1/64 \log(\rho_2)$, hence for large $\mu$, $\mu \geq \mu_3$, we have

$$C_3 C_4 \rho_2^k \leq \exp(-\delta_1(64\mu)).$$

Now, if $\mu \geq \mu_4$ is large enough and using $|C_1| = O(\mu^2)$, we can write

$$|S_1(t)| \leq C_1 C_3 C_4 \rho_2^k \leq \exp(-\delta_1(32\mu)), \quad t \in H.$$ 

Integrating this on $[a_1, x]$, $x \leq \alpha - \rho_1$, we obtain for large $\mu$, $\mu \geq \mu_5$, that

$$|S(x)| = \left| \int_{a_1}^x S_1(t) dt \right| \leq c_2 C_1 C_3 C_4 \rho_2^k \leq \exp(-\delta_1(16\mu))$$

moreover this also holds when $x \in [a_0, a_1]$. If $x \in [\alpha + \rho_1, x_0]$, then using that $S_1(t) \geq 0$ when $t \in [\alpha + \rho_1, x_0]$, we can write

$$1 - \exp(-\delta_1(16\mu)) \leq 1 - c_2 C_1 C_3 C_4 \rho_2^k \leq \int_{a_1}^{x_0} S_1(t) dt - \int_{x}^{x_0} S_1(t) dt$$

$$= S(x) - S(x_0) = 1.$$

Similarly when $x \in [x_0, \beta - \rho_1]$, $S_1(t) \leq 0$ on $[x_0, \beta - \rho_1]$, hence

$$1 - \exp(-\delta_1(16\mu)) \leq 1 - c_2 C_1 C_3 C_4 \rho_2^k \leq \int_{a_1}^{x_0} S_1(t) dt + \int_{x}^{x_0} S_1(t) dt$$

$$= S(x) - S(x_0) = 1.$$ 

As for $[\beta + \rho_1, a_{l+1}]$, we know that $|S_1(t)| \leq C_1 C_3 C_4 \rho_2^k \leq \exp(-\delta_1(32\mu))$, and $S(a_{l+1}) = 0$, so for $x \in [a_{l+1}, \alpha_{l+1}]$, $S(x) = \int_{a_1}^x S_1(t) dt = \int_{a_1}^{a_{l+1}} S_1(t) dt$ and $|S(x)| \leq c_2 C_1 C_3 C_4 \rho_2^k \leq \exp(-\delta_1(16\mu))$. For $x \in [\beta + \rho_1, \alpha_{l+1}]$, we know that

$$S(x) = \int_{a_1}^{x} S_1(t) dt = \int_{a_1}^{a_{l+1}} S_1(t) dt - \int_{x}^{a_{l+1}} S_1(t) dt$$

$$= 0 + \int_{x}^{a_{l+1}} -S_1(t) dt = \int_{x}^{a_{l+1}} |S_1(t)| dt \leq c_2 C_1 C_3 C_4 \rho_2^k \leq \exp(-\delta_1(16\mu)).$$
These last four displayed estimates show that (8) and first half of (9) hold since 
\[ \exp(-\delta_1(16\mu)) \leq \exp(-2\delta_1(3\deg S)) \]
if \( \mu \geq \mu_6 \) is large. (10) and (7) are also true, since \( S'(\cdot) = S_1(\cdot) \) is nonnegative on \((a_{l_0}, x_0)\) and is nonpositive on \((x_0, a_{l_0+1})\).

To establish the second half of (9) (with \( Z(x) \) in place of \( \min(1, |Z(x)|) \)), we write (similarly to \( 18 \))
\[
|S(x)| = \left| C_1 \int_{a_1}^x Z_1(t) \frac{P_1(t)}{||P_1||_H} R(t)(t-x_0)^{k_0'} dt \right| \|P_1\|_H \\
\leq |C_1| \int_{a_1}^x Z_1(t) \frac{|P_1(t)|}{||P_1||_H} ||R(t)|| |t-x_0|^{k_0'} dt \|P_1\|_H \\
\leq |C_1| C_4 \int_{a_j}^x Z_1(t) dt \|P_1\|_H
\]
where \( x \in [a_j, a_{j+1}] \) and \( H = [a_0, \alpha - \rho_1] \cup [\alpha + \rho_1, \beta - \rho_1] \cup [\beta + \rho_1, a_{l+1}] \). It is easy to see that
\[
\frac{\int_{a_j}^x Z_1(t) dt}{|Z(x)|}
\]
has finite limit as \( x \to a_j \) since \( Z \) and \( Z_1 \) have zeros of order \( k_j \) and \( k_j' \) at \( a \) respectively. The same is true on the left hand side neighborhood of \( a_j \). Hence we see that \( \int_{a_1}^x Z_1(t) dt / |Z(x)| \) is bounded when \( x \in H \), so, using \( \|P_1\|_H \leq \exp(-\delta_1 32\mu) \) coming from \( 19 \), we obtain that the second half of (9) holds for large \( \mu, \mu \geq \mu_7 \).

To fulfill (12), consider \( Q := S^2 \). Then, the degree of \( Q \) is \( 2(k_1' + \ldots + k_l' + l - 2 + k_0' + 2\mu + 1) = 4\mu + \text{const} \). By squaring \( S \) defined in (13), it is easy to see that (5), (6), (7), (9), (11) and (10) are preserved, and actually, (8) too:
\[
(1 - \exp(-2\delta_1(3\deg S)))^2 \geq 1 - \exp(-2\delta_1 \deg Q)
\]
since \( 2 \exp(-2\delta_1 3 \deg S) - \exp(-4\delta_1 3 \deg S) \leq \exp(-2\delta_1 \deg Q) \) if \( \deg S \) is large (that is, if \( \mu \geq \mu_8 \)).

Finally, we have a sequence of polynomials for particular degrees. The basic idea to use the same polynomial for larger degree works now, because of the following. Put \( m_1(m) := \max\{m_1 : m_1 = 4\mu + 2(k_1' + \ldots + k_l' + l - 2 + k_0' + 1) \}, m_1 \leq m, \mu \in \mathbb{N} \}. For general \( m \in \mathbb{N} \), replacing the error term for \( m \) from \( m_1(m) \) brings in a factor \( \exp(-2\delta_1 m)/\exp(-2\delta_1 m_1(m)) \) which can be estimated as
\[
\limsup_{m \to \infty} \exp(-2\delta_1 m)/\exp(-2\delta_1 m_1(m)) = \exp(-2\delta_1 \text{const}) < 1,
\]
where \( \text{const} \) is actually \( 2(k_1' + \ldots + k_l' + l - 2 + k_0' + 1) \). Hence, if \( \mu \geq \mu_9 \) is large, then
\[
\exp(-2\delta_1 m_1(\deg Q)) \leq \exp(-\delta_1 \deg Q)
\]
which finishes the proof.
Remark: Note that (the second half of) (9) implies (11).

We need the following trigonometric form of fast decreasing polynomials. In the proof we use so-called half-integer trigonometric polynomials

\[ \sum_{j=0}^{\infty} a_j \cos((j+1/2)t) + b_j \sin((j+1/2)t). \]

They are natural in this context, see, e.g., the product representation [2], p. 10, or Videnskii’s original paper [30], or the paper [16].

**Theorem 3.** Let \( t_0, \alpha, \beta, \alpha', \beta' \in (-\pi, \pi) \) be such that

\[-\pi < \alpha' < \alpha < t_0 < \beta < \beta' < \pi \quad \text{and} \quad \alpha_1, \ldots, \alpha_l \in (-\pi, \pi) \setminus [\alpha', \beta'] \]

with the corresponding positive integer powers \( k_1, \ldots, k_l \). Put

\[ Z(t) := \prod_{j=1}^{l} \left| \sin \frac{t - \alpha_j}{2} \right|^{k_j}. \]

Then there exists \( \delta_1 > 0 \) such that for all large \( m \) there exists a trigonometric polynomial \( Q_m \) with degree at most \( m \) such that

\[ Q_m(t_0) = 1, \]

\[ 0 \leq Q_m(t) < 1 \quad \text{for} \quad t \in [-\pi, \pi), t \neq t_0, \]

\[ Q_m^{(k_j)}(\alpha_j) = 0, \quad j = 1, \ldots, l, \quad k = 0, 1, \ldots, k_j, \]

\[ Q_m(t) \leq \min(1, |Z(t)|) \exp(-\delta_1 m) \quad \text{for} \quad t \in [-\pi, \pi] \setminus [\alpha', \beta'], \]

\[ |Q_m(t) - 1| \leq \exp(-\delta_1 m) \quad \text{for} \quad t \in [\alpha, \beta], \]

\[ Q_m(t) \text{ is strictly monotone on } [\alpha', \alpha] \text{ and on } [\beta, \beta']. \]

**Proof.** Briefly, we use similar idea as in the previous proof (Theorem 2), but there are lots of differences.

First, we introduce the intervals between the neighboring \( \alpha_j \)’s as follows using the ordering of \( \alpha_j + \epsilon_j 2\pi, \quad j = 1, \ldots, l \), and \( \epsilon_j = 0 \) if \( \alpha_j > \beta' \) and \( \epsilon_j = 1 \) otherwise. Let \( I_j \)’s, \( j = 1, \ldots, l - 1 \) denote the closed intervals such that endpoints are the \( \alpha_j + \epsilon_j 2\pi \)’s and they are disjoint except for the endpoints, and they are ordered from left to right (that is, if \( t_1 \in I_j \) and \( t_2 \in I_k \) and \( j \leq k \), then \( t_1 \leq t_2 \)). Denote the left endpoint of \( I_1 \) by \( \alpha_* \), and the right endpoint of \( I_{l-1} \) by \( \alpha^* \), that is, \( \alpha_* \) and \( \alpha^* \) are the minimum and maximum of \( \alpha_j + \epsilon_j 2\pi \)’s respectively. Put \( I_0 := [\alpha^* - 2\pi, \alpha_*] \), this way \( I_0, I_1, \ldots, I_{l-1} \) cover an interval of length \( 2\pi \) and \( t_0 \in I_0, [\alpha', \beta'] \subset I_0 \). Note that \( I_j \)’s are not necessarily subsets of \( (-\pi, \pi) \).
We define
\[
\tilde{Z}_1(t) := \prod_{j=1}^l \left( \sin \frac{t - \alpha_j}{2} \right)^{k'_j}, \quad \tilde{R}(\tau; t) := \tilde{R}(t) := \prod_{j=1}^{l-1} \sin \frac{t - \tau_j}{2},
\]
\[
\tilde{P}_0(t) = \tilde{P}_0(a, \mu; t) := \left( \cos \frac{t - a}{2} \right)^{2\mu},
\]
\[
\tilde{P}_1(a, b, \lambda, \mu; t) = (1 - \lambda)\tilde{P}_0(a, \mu; t) + \lambda \tilde{P}_0(b, \mu; t)
\]
where \( k'_j = k_j \) if \( k_j \) is even and \( k'_j = k_j + 1 \) if \( k_j \) is odd, for \( j = 1, \ldots, l \), and \( \tau_j \in I_j, \ j = 1, \ldots, l - 1 \), and \( \alpha' < a < \alpha < \beta < \beta' \), \( a := (\alpha + \alpha')/2 \), \( b := (\beta + \beta')/2 \), and \( \lambda \in [0, 1] \). We also put \( k'_0 = k_0 \) if \( k_0 \) is odd and \( k'_0 = k_0 + 1 \) if \( k_0 \) is even; and \( \sigma := (\tau_1, \ldots, \tau_{l-1}) \). As above, if some of the parameters are fixed or unimportant in the current consideration, then we leave them out, e.g. \( \tilde{P}_0(t) = \tilde{P}_0(a, \mu; t) \) and \( \tilde{P}_1(t) = \tilde{P}_1(\mu; t) = \tilde{P}_1(\lambda, \mu; t) = \tilde{P}_1(a, b, \lambda, \mu; t) \).

Some immediate properties are the following: \( \tilde{Z}(t) \), \( \tilde{P}_0(t) \) and \( \tilde{P}_1(t) \) are nonnegative trigonometric polynomials. If \( l \) is even, then \( \tilde{R}(t) \) is a half-integer trigonometric polynomial, if \( l \) is odd, then it is a trigonometric polynomial (with degree \((l - 1)/2\)).

Consider
\[
\tilde{S}_1(t) := \tilde{Z}_1(t) \tilde{P}_1(\mu; t) \tilde{R}(\tau; t) \left( \sin \frac{t - t_0}{2} \right)^{k'_0}
\]
which is a trigonometric polynomial if \( l \) is even and is a half-integer trigonometric polynomial if \( l \) is odd. We need
\[
\tilde{S}_2(t) := \begin{cases}
\tilde{S}_1(t), & \text{if } l \text{ is even}, \\
\tilde{S}_1(t) \cos \frac{t - (\alpha^* - \pi)}{2}, & \text{if } l \text{ is odd}
\end{cases}
\]
which is a trigonometric polynomial in both cases.

Now we would like to integrate \( \tilde{S}_1(\cdot) \) and get a trigonometric polynomial too. To do this, we use Poincaré-Miranda theorem, as in the proof of Theorem 2. Consider the rectangle \( R := [0, 1] \times I_1 \times I_2 \times \ldots \times I_{l-1} \) and \( (\lambda, \tau) = (\lambda, \tau_1, \ldots, \tau_{l-1}) \in R \). We use the functions
\[
f_j(\lambda, \tau; \mu; t) := \int_{I_j} \tilde{S}_2(\lambda, \tau; \mu; t) \, dt, \quad j = 0, 1, \ldots, l - 1.
\]
Note that \( \sin \frac{t - t_0}{2} \) is negative on \( (\alpha^* - 2\pi, t_0) \) and is positive on \( (t_0, \alpha^*) \), \( \cos \frac{t - (\alpha^* - \pi)}{2} \) is positive on \( (\alpha^* + 2\pi, \alpha^*) \) but it introduces an extra zero at \( \alpha^* \). It can be verified same way as in the proof of Theorem 2 that there are sign changes in \( f_0 \) as \( \lambda \) changes from 0 to 1, and in \( f_j \) as \( \tau_j \) goes from the left endpoint of \( I_j \) to the right endpoint of \( I_j \).

Poincaré-Miranda theorem shows that there are particular \( \lambda \in [0, 1], \tau_1 \in I_1, \ldots, \tau_{l-1} \in I_{l-1} \) such that all the \( f_j \)’s are zero; fix this solution and denote
it by $\lambda, \tau_1, \ldots, \tau_{l-1}$ in the rest of this proof. Summing up these integrals for all $j = 0, 1, \ldots, l-1$, we also obtain that $\int_{\alpha^*-2\pi}^{\alpha^*} \tilde{S}_2(t) dt = 0$.

Put
\[
\tilde{S}(t) := \int_{\alpha^*}^t C_1 \tilde{S}_2(\tau) d\tau
\]
where $C_1$ will be chosen later (like in the proof of Theorem 2). In both cases ($l$ is even or odd), the integrand is a real trigonometric polynomial. Since the integral of $\tilde{S}_2(t)$ over $[\alpha^*-2\pi, \alpha^*]$ is 0, $\tilde{S}(t)$ is also a trigonometric polynomial. $C_1$ can be chosen so that
\[
\int_{\alpha^*}^{t_0} C_1 \tilde{S}_2(t) dt = 1
\]
holds. The properties (20), (22), (23), (24) and (25) can be verified same way as in the proof of Theorem 2. A key tool was the Nikolskii inequality for algebraic polynomials and it should be replaced with the similar inequality for trigonometric polynomials, which is again due to Nikolskii (see, e.g [14], p. 495, Theorem 3.1.1). Again, squaring $\tilde{S}$, we can construct the trigonometric polynomial which also satisfies (21).

### 3.2 Rough Markov- and Bernstein-type inequalities

The following two propositions have rather simple proofs, they may be known, but we could not find reference for them.

**Proposition 4.** Let $I \subset (-\pi, \pi)$ be a closed set consisting of finitely many disjoint intervals such that none of them is a singleton and $k$ be a positive integer. Then there exists $C = C(I, k) > 0$ such that for all trigonometric polynomial $T_n$ with degree $n$, we have
\[
\left\| T_n^{(k)} \right\|_I \leq C n^k \| T_n \|_I .
\] (26)

This immediately follows from iterating Videnskii’s inequality on each component (maximal subinterval) of $I$. For Videnskii’s inequality, see [2], p. 243 (Exercise E.19 part c)] or [31].

We also need a rough Bernstein-type inequality for higher derivatives of trigonometric polynomials.

**Proposition 5.** Let $I \subset (-\pi, \pi)$ be again a closed set consisting of finitely many disjoint intervals such that none of them is a singleton and $k$ be a positive integer. Fix a closed set $I_0 \subset \text{Int } I$ (subset of the one dimensional interior of $I$). Then there exists $C = C(I, I_0, k) > 0$ such that for all trigonometric polynomial $T_n$ with degree $n$, we have for $t \in I_0$
\[
\left| T_n^{(k)}(t) \right| \leq C n^k \| T_n \|_I .
\] (27)

This again, immediately follows from applying Videnskii’s inequality (see [2], p. 243, E.19 part b)] iteratively on the component (say $I_n^+$) of $I$ containing $I_0$ and finally using $\| T_n \|_{I_0^+} \leq \| T_n \|_I$. 

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3.3 Asymptotically sharp Markov-type inequality

Theorem 6. Let \( E \subset (-\pi, \pi) \) be a compact set satisfying (3). Then for any trigonometric polynomial \( T_n \) with degree \( n \), we have

\[
\left\| T_n^{(k)} \right\|_{[a-\rho, a]} \leq (1 + o(1))n^{2k} \Omega(E_T, e^{ia})^{2k} \frac{8^k \pi^{2k}}{(2k-1)!!} \left\| T_n \right\|_E \tag{28}
\]

where \( o(1) \) is an error term that tends to \( 0 \) as \( n \to \infty \), depends on \( E \) and \( a \), but it is independent of \( T_n \). This inequality is sharp, that is, there is a sequence of trigonometric polynomials \( T_n, \ n = 1, 2, \ldots \), such that \( \text{deg} T_n = n \) and

\[
\left| T_n^{(k)}(a) \right| \geq (1 - o(1))n^{2k} \Omega(E_T, e^{ia})^{2k} \frac{8^k \pi^{2k}}{(2k-1)!!} \left\| T_n \right\|_E , \tag{29}
\]

where \( o(1) \to 0 \) is an error term depending on \( E \) and \( n \).

Proof. The proof of (28) is divided into five steps and then (29) will be established.

First step. We prove the assertion when \( E \) is a T-set, and \( T_n \) is polynomial of the defining polynomial \( U_N \) for this set. That is, \( E = \{ t \in (-\pi, \pi) : |U_N(t)| \leq 1 \} \) (as in (1)) and there is a real, algebraic polynomial \( P \) such that \( T_n(t) = P(U_N(t)) \). We may assume that \( U_N(a) = 1 \) (we know that \( |U_N(a)| = 1 \)).

Now we use Faà di Bruno’s formula (1). Note that, in our setting \( f = P \) (outer function) and \( g = U_N \) (inner function), hence the product is independent of \( P \) and \( n \) (and \( T_n \) too). Hence we reorder the terms decreasingly:

\[
(P \circ U_N)^{(k)}(a) = P^{(k)}(1)(U_N'(a))^k + \ldots \tag{30}
\]

where in the remaining terms only \( P^{(k-1)}(1), \ldots, P'(1) \) occur. There are finitely many remaining terms and by (28), they grow like \( n^{2k-2} \) as \( n \to \infty \).

As for the first term, we can use the classical V. Markov inequality (see e.g. [2], p. 254) and \( \|P\|_{[-1,1]} = \left\| T_n \right\|_E \), hence with \( d := \text{deg}(P) \),

\[
|P'(1)| \leq \frac{d^2(d^2 - 1) \ldots (d^2 - (k - 1)^2)}{(2k - 1)!!} \left\| T_n \right\|_E \leq \frac{d^{2k}}{(2k - 1)!!} \frac{1}{\left\| T_n \right\|_E}
\]

where actually

\[
\frac{d^2(d^2 - 1) \ldots (d^2 - (k - 1)^2)}{d^{2k}} \to 1 \tag{31}
\]

as \( n \to \infty \) (which is equivalent to \( d \to \infty \)).

As for \( U_n'(a) \), we use the density of the equilibrium measure, more precisely formula (3.21) from [28] (and \( a = a_{2j_0} \)), hence

\[
|U_n'(a)| = 2N^2 \prod_{i=1}^{m} \frac{|e^{ia} - e^{i\tau_i}|^2}{|e^{ia} - e^{ia}|} = 2N^2 \prod_{i=1}^{m} \frac{|e^{ia} - e^{i\tau_i}|^2}{|e^{ia} - e^{ia}|} = 2N^2 M_{a,k} = 8\pi^2 N^2 \Omega(E_T, e^{ia})^2 . \tag{32}
\]
Putting these together:
\[
|T_n^{(k)}(a)| \leq (1 + o(1))8^k \pi^{2k} \frac{1}{(2k - 1)!} \Omega(E_T, e^{i\alpha})^{2k} n^{2k} \|T_n\|_E.
\]

Now we extend the previous inequality from \([a - \rho, a]\) (as in (28)) to \([a - \eta, a]\) where \(\eta > 0\).

Basicly we use the smaller growth of the rough Bernstein-type inequality (27) and the continuity of \(U'_N\). For any \(\varepsilon > 0\), we can select \(\eta > 0\) such that \([a - \eta, a] \subset E\) and for \(t \in [a - \eta, a]\) it is true that
\[
|U'_N(t)| \leq (1 + \varepsilon)|U'_N(a)| = (1 + \varepsilon)8\pi^2 N^2 \Omega(E_T, e^{i\alpha})^2.
\]

Then for \(t \in [a - \eta, a]\) we get from (30) and again from (26) that
\[
|T_n^{(k)}(t)| \leq (1 + o(1))(1 + \varepsilon)8^k \pi^{2k} \frac{1}{(2k - 1)!} \Omega(E_T, e^{i\alpha})^{2k} n^{2k} \|T_n\|_E. \tag{33}
\]

Now, on \([a - \rho, a - \eta]\) (if not empty), we can use the rough Bernstein-type inequality (27), hence we obtain an upper estimate for \(T^{(k)}(t)\) which has growth order \(n^k\), which is smaller than \(n^{2k}\), the growth order of the Markov factor. So if \(n\) is large (depending on \(\varepsilon\)), then (33) holds for \(t \in [a - \rho, a - \eta]\) too. Now letting \(\varepsilon \to 0\) appropriately, (28) follows for \(T_n(.) = P(U_N(.))\) as \(d = \deg(P) \to \infty\).

Second step. Now we establish (28) when \(E\) is a T-set and \(T_n\) is arbitrary trigonometric polynomial. We use symmetrization here (see, 25, pp. 151-152 and 28, pp. 2997-2998, including Lemma 3.2) and fast decreasing trigonometric polynomials (see Subsection 3.1). In this step we work in a smaller neighborhood of \(a\), i.e. on \([a - \rho_0, a]\) where \(\rho_0 < \rho\) is defined later.

Let \(j_0\) correspond to the interval in which \(a\) is. More precisely, since \(E\) is a T-set in this case, there are \(2N\) disjoint, open intervals such that \(U_N\) maps these intervals to \((-1, 1)\) in a bijective way. Let us label them by \(E_j = (\alpha_{2j-1}, \alpha_{2j})\) where \(-\pi < \alpha_1 < \alpha_2 \leq \alpha_3 < \alpha_4 \leq \ldots \leq \alpha_{2N-1} < \alpha_{2N} < \pi\). Hence \(a \in [\alpha_{2j_0-1}, \alpha_{2j_0}]\) and by (2), \(a = \alpha_{2j_0}\). Put \(\rho_0 := 1/4\min(\alpha_{2j_0} - \alpha_{2j_0-1}, \alpha_{2j_0+1} - \alpha_{2j_0}, \rho, \pi/4)\).

We also need the following facts on T-sets. Since \(U_N(.)\) is \(2N\)-to-1 mapping, we need its restricted inverses. Let \(U_{N,j}^{-1}(t)\) be the inverse of \(U_N\) restricted to \([\alpha_{2j-1}, \alpha_{2j}]\) and put \(t_j(t) = U_{N,j}^{-1}(U_N(t))\). Obviously, \(t_j\) is \(C^\infty\) on \(\cup_{j=1}^N(\alpha_{2j-1}, \alpha_{2j})\) and now we give estimates for the \(l\)-th derivative of \(t_j(t)\), especially, as \(t\) approaches \(a\). Similarly, as in (23), if \(l = 1\) or \(l = 2\), then
\[
\begin{align*}
\frac{dt_j}{dt} &= \frac{d}{dt} U_{N,j}^{-1}(U_N(t)) = \frac{U'_N(t)}{U_N(U_{N,j}^{-1}(U_N(t)))} = \frac{U'_N(t)}{U_N(t)}, \\
\frac{d^2}{dt^2} U_{N,j}^{-1}(U_N(t)) &= \frac{-(U'_N(t))^2 U''_N(U_{N,j}^{-1}(U_N(t))))}{(U_N(U_{N,j}^{-1}(U_N(t))))^3} + \frac{U''_N(t)}{U_N(U_{N,j}^{-1}(U_N(t)))} \\
&= -\left(\frac{U''_N(t)^2 U''_N(t_j)}{U_N(t_j)^3}ight) + \frac{U''_N(t)}{U_N(t_j)}.
\end{align*}
\]
and for general $l$, Faà di Bruno’s formula \[ \prod \] implies that there is a universal polynomial $Q_l$ (dependent on $U_N$, depending on $l$ only) which is a polynomial in $U_N^{(k)}(t)$ and $U_N^{(l)}(t_j) \; k, l = 1, \ldots, l$, that is $Q_l = Q_l(U_N^{(k)}(t), \ldots, U_N^{(l)}(t_j), \ldots)$ such that
\[
\frac{d^l}{dt^l} U_N^{-1}(U_N(t)) = \frac{Q_l}{(U_N'(t_j))^{2l-1}}. \tag{34}
\]
Here, $Q_l$ is independent of $n$ and $T_n$, hence $|Q_l| \leq C$ for some $C = C(k, U_N) > 0$.

Moreover, we need to estimate $|U_N'(t_j)|$ as $t \to a$ and we split the argument into two cases. If $j$ is such that $a_j \in \text{Int} E$, that is, $U_N'(a_j) = 0$, and using that all the zeros of $U_N$ are simple, we can infer that $U_N''(a_j) \neq 0$, so $|U_N'(t_j)| \geq O(|t_j - a_j|)$. On the other hand, if $j$ is such that $a_j \in E \setminus \text{Int} E$, that is, $U_N'(a_j) \neq 0$, then simply $U_N'(t_j) \approx U_N'(a_j)$. Hence, in any case
\[ |U_N'(t_j)| \geq O(|t_j - a_j|). \tag{35} \]

For an arbitrary polynomial $T_n$ consider $V_n(t) = L_{\sqrt{\pi}}(t) T_n(t)$, where $L_{\sqrt{\pi}}(.)$ denotes the fast decreasing polynomial which has the following properties. $L_{\sqrt{\pi}}(.)$ has degree at most $\sqrt{\pi}$, it is a fast decreasing trigonometric polynomial and peaking at $a$ very smoothly (that is, $L_{\sqrt{\pi}}(a) = 1$ and $L_{\sqrt{\pi}}''(a) = 0$, $j = 1, 2, \ldots, 2k^2$). $L_{\sqrt{\pi}}(.)$ is approximately 1 on $[a - \rho_0, a + \rho_0]$ and is approximately 0 outside $[a - 2\rho_0, a + 2\rho_0]$ and vanishes at the other extremal points of $U_N$ up to order $2k^2$ (that is, if $U_N(t) = \pm 1$, $t \neq a$, then $L_{\sqrt{\pi}}''(t) = 0$, $j = 0, 1, \ldots, 2k^2$). Such polynomial $L(.) = L_{\sqrt{\pi}}(.)$ exists because of Theorem \[ \prod \].

For simplicity, put $W(t) := \prod_j \left( \sin \frac{t - \alpha_j}{2} \right)^{2k}$ where $j = 1, \ldots, 2N, j \neq j_0$. This $W$ is a nonnegative trigonometric polynomial and has sup norm at most 1. There is another trigonometric polynomial $Y(.)$ such that
\[ L(t) = Y(t)W^k(t). \]
The sup norm of $Y$ over $[-\pi, a - \rho_0] \cup [a + \rho_0, \pi]$ can be estimated using \[ \prod \] with $W^k$ in place of $Z$. Hence, for $t \in [-\pi, a - \rho_0] \cup [a + \rho_0, \pi]$
\[ |Y(t)| = \left| \frac{L(t)}{W^k(t)} \right| \leq \min \left( \frac{1}{W^k(t)}, 1 \right) \exp (-\text{deg} L) \delta_1. \]

Differentiating $L(.)$ $j$-times, $j = 0, 1, \ldots, k$ we write
\[ L^{(j)}(t) = \sum_{l=0}^j \binom{j}{l} Y^{(j-l)}(t) (W^k)^{(l)}(t). \tag{36} \]
Here $(W^k)^{(l)}(t) = W(t) \cdot \ldots$ where $W(t)$ is multiplied with other terms depending on $W, W', \ldots, W^{(l)}, k$ and $\alpha_j$’s only, and it is independent from $n$ and $T_n$. As regards $Y^{(j-l)}(t)$, we can use Videnskii’s inequality for $Y(.)$ on
These imply that for all \(t \in [-\pi, a - 2\rho_0] \cup [a + 2\rho_0, \pi]\) and all \(l = 0, 1, \ldots, j\) \n
\[
   \left| Y^{(j-l)}(t) \right| \leq C (\deg Y)^{j-l} \exp \left( -(\deg L)\delta_1 \right) .
\]  

Summing up these estimates as in \([30]\), we can write with \(\deg L \leq \sqrt{n}\)

\[
   \left| L^{(j)}(t) \right| \leq CW(t)n^j/2 \exp \left( -\sqrt{n}\delta_1 \right)
\]

where \(C > 0\) is independent of \(n\) and \(T_n\), and \(t \in [-\pi, a - 2\rho_0] \cup [a + 2\rho_0, \pi]\).

This \(V_n\) has degree at most \(n + \sqrt{n}\) and satisfies

\[
\begin{aligned}
   &\|V_n\|_E \leq \|T_n\|_E , \\
   &V_n(t) = \left(1 + O(\beta\sqrt{n^2})\right)T_n(t) \text{ for } t \in [a - \rho_0, a], \\
   &|V_n(t)| = O(\beta\sqrt{n^2}) \|T_n\|_E \text{ for } t \in E \setminus [a - 2\rho_0, a]
\end{aligned}
\]

where \(\beta = \exp(-\delta_1) < 1\).

Now, (by Leibniz formula), for all \(l = 1, \ldots, k\)

\[
   V_n^{(l)}(t) - T_n^{(l)}(t) = (L_{\sqrt{\pi}}(t) - 1)T_n^{(l)}(t) + \sum_{j=1}^{l} \binom{l}{j} L_{\sqrt{\pi}}^{(j)}(t)T_n^{(l-j)}(t).
\]

Using the rough Markov-type inequality \([26]\), there exists a constant \(C = C(E, k) > 0\) such that for all \(1 \leq j \leq k, t \in E\)

\[
\begin{aligned}
   &\left| L_{\sqrt{\pi}}^{(j)}(t) \right| \leq C\sqrt{n}^j \|L_{\sqrt{\pi}}\|_E = Cn^j, \\
   &\left| T_n^{(j)}(t) \right| \leq Cn^j \|T_n\|_E
\end{aligned}
\]

and if \(t \in E \setminus [a - 2\rho, a]\), then applying \([26]\) for \(L_{\sqrt{\pi}}\) on \(E \setminus [a - 2\rho, a]\), we can write

\[
   \left| L_{\sqrt{\pi}}^{(j)}(t) \right| \leq C\sqrt{n}^j \|L_{\sqrt{\pi}}\|_{E \setminus [a - 2\rho, a]} = Cn^j \beta\sqrt{n}.
\]

These imply that for \(l = 1, \ldots, k\)

\[
   \left| V_n^{(l)}(t) - T_n^{(l)}(t) \right| = O \left( n^{2l}\beta\sqrt{n} + n^{2l-1}\right) \|T_n\|_E, \ t \in [a - \rho, a]
\]

and

\[
   \left| V_n^{(l)}(t) \right| = O \left( n^{2l}\beta\sqrt{n}\right) \|T_n\|_E, \ t \in E \setminus [a - 2\rho, a].
\]

Define the "symmetrized" polynomial as

\[
   T^*(t) := \sum_{j=1}^{N} V_n(t_j).
\]
This $T^*$ will be algebraic polynomial of $U_N(\cdot)$, see Lemma 3.2 in [28], and deg($T^*$) $\leq n + \sqrt{n} = (1 + o(1))n$.

Now we compare $(T^*)^{(k)}(t)$ with $T_n^{(k)}(t)$ when $t \in [a - \rho, a]$. If $j = j_0$, that is, $t_j = t$, then $V_n(t_j) = V_n(t)$, and we can apply [42] (when $l = k$). If $j \neq j_0$, then we would like to show that $\left| \frac{d^k}{dt^k} V_n(t_j) \right|$ is small. We use [40] first so

$$\left| \frac{d^k}{dt^k} V_n(t_j) \right| \leq \sum_{l=0}^{k} \binom{k}{l} \left| \frac{d^l}{dt^l} L \sqrt{n} \left( U^{-1}_{N,j}(U_N(t)) \right) \right|^k \left| \frac{d^{k-l}}{dt^{k-l}} T_n \left( U^{-1}_{N,j}(U_N(t)) \right) \right|$$

![Equation](image)

which we continue later. For the second factor, we use [1] again with similar groupings of the terms as in [30], because the first term involves $\frac{d^{k-l}}{dt^{k-l}} T_n$ (at $t_j$) and all the other terms involve lower derivatives of $T_n$. So we can write, with the help of [26], and [34], [35]

$$\left| \frac{d^{k-l}}{dt^{k-l}} T_n \left( U^{-1}_{N,j}(U_N(t)) \right) \right| \leq \left| T_n^{(k-l)}(t_j) \right| \left| \frac{d}{dt} U^{-1}_{N,j}(U_N(t)) \right|^{k-l} + \ldots$$

$$\leq C n^{2k-2l} \frac{1}{|t_j - a_j|^{2(k-l)-1}} \|T_n\|_E .$$

Now we use the zeros of $L \sqrt{n}(\cdot)$ (and $W(t)$) to get rid of the factors $1/|t_j - a_j|^{2(k-l)-1}$.

To estimate the first factor on rhs of (43), we use [1] for $L \sqrt{n}(\cdot)$ and $U^{-1}_{N,j}(U_N(t))$ with [11] (since $t_j \notin [a - 2\rho, a]$) and [38]. Hence

$$\left| \frac{d}{dt} L \sqrt{n} \left( U^{-1}_{N,j}(U_N(t)) \right) \right| \leq C \sqrt{n} 2^{j_l} \beta \sqrt{n} |W(t_j)| n^{2l} \frac{1}{|t_j - a_j|^{2l-1}} \|T_n\|_E$$

$$= C n^l \beta \sqrt{n} \frac{|W(t_j)|}{|t_j - a_j|^{2l-1}} \|T_n\|_E$$

and using that $a_j$ is a zero of $W$ (of order $k$), the fraction $|W(t_j)|/|t_j - a_j|^{2l-1}$ is actually bounded.

Multiplying together the last two displayed estimates and using that $|W(t_j)|/|t_j - a_j|^{2k}$ is bounded (independently of $t$, $j$ and $n$), we can continue (43),

$$\leq \sum_{l=0}^{k} \binom{k}{l} C n^l \beta \sqrt{n} n^{2k-2l} \|T_n\|_E \leq C n^{2k} \beta \sqrt{n} \|T_n\|_E .$$

Collecting all the calculations in this paragraph, for $t \in [a - \rho, a]$ we can write

$$\left| (T^*)^{(k)}(t) - T_n^{(k)}(t) \right| \leq O \left( n^{2k} \beta \sqrt{n} \right) \|T_n\|_E .$$

![Equation](image)

Comparing the sup norms of $T_n$ and $T^*$, we split the estimate into two cases (see also [39]). If $t \in E \setminus [a - 2\rho_0, a]$, then

$$|T^*(t)| \leq \sum_{j=1}^{N} |L \sqrt{n}(t_j)| |T_n(t_j)| \leq NC \beta \sqrt{n} \|T_n\|_E = o(1) \|T_n\|_E .$$

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If \( t \in [a - 2\rho_0, a] \), then

\[
|T^*(t)| \leq |L_{\sqrt{n}}(t)| T_n(t) + \sum_{j \neq j_0} |L_{\sqrt{n}}(t_j)| |T_n(t_j)|
\]

\[
\leq \left(1 + NC\beta n\right) \|T_n\|_E = (1 + o(1)) \|T_n\|_E.
\]

These two estimates yield

\[
\|T^*\|_E \leq (1 + o(1)) \|T_n\|_E. \tag{45}
\]

Applying (44), (45) and the previous case for \( T^* \) (when \( T^* \) is a polynomial of \( U_N \)), we obtain (28) for \( T \)-sets and for arbitrary polynomials.

Third step. Now let \( E \) be an arbitrary set consisting of finite number of intervals: \( E = \cup_{j=1}^m [a_{2j-1}, a_{2j}] \). Using the density of \( T \)-sets (see Section 2.1), there is a \( T \)-set \( E' \) such that \( E' \subset E \), \( a \in E' \) and

\[
\Omega(E_T, e^{i\alpha}) \leq \Omega(E'_T, e^{i\alpha}) \leq (1 + \varepsilon)\Omega(E_T, e^{i\alpha})
\]

where \( \varepsilon > 0 \) is arbitrary and \( E' = E'(E, \varepsilon) \). Here the first inequality comes from the monotonicity of \( \Omega(., .) \) (and from \( E' \subset E \)) and the second comes from the density result. Obviously, \( \|T_n\|_{E'} \leq \|T_n\|_E \). Now, applying the previous step (for arbitrary polynomials on \( T \)-sets), we can write for \( t \in [a - \rho, a] \)

\[
|T^{(k)}_n(t)| \leq (1 + o_{E'}(1)) \frac{8^k \pi^{-2k}}{(2k - 1)!} n^{2k} \Omega(E'_T, e^{i\alpha})^{2k} \|T_n\|_{E'}
\]

\[
\leq (1 + o_E(1)) \frac{8^k \pi^{-2k}}{(2k - 1)!} n^{2k} \Omega(E_T, e^{i\alpha})^{2k} \|T_n\|_E
\]

by letting \( \varepsilon \to 0 \) appropriately.

Fourth step. Now let \( E \subset (-\pi, \pi) \) be a compact set which is regular (in the sense of Dirichlet problem). Obviously, the regularity of \( E \) and \( E_T \) are equivalent.

Consider the trigonometric polynomial \( T_nQ_{\nu_0} \) of degree at most \( n(1 + \varepsilon) \) where \( Q(.) = Q_{\nu_0}(.) \) is the fast decreasing polynomial with the following properties: its degree is at most \( n\varepsilon \), \( 0 \leq Q(.) \leq 1 \), \( Q(t) \leq \exp(-\delta_1 n\varepsilon) \) for some \( \delta_1 > 0 \) on \( t \in [-\pi, a - 2\rho] \cup [a + 2\rho, \pi] \), \( 1 - \exp(-\delta_1 n\varepsilon) \leq Q(t) \) on \( t \in [a - \rho, a + \rho] \) and \( Q(a) = 1 \) (for existence, see Section 3.1).

Let \( g_{E_T}(\zeta, 0) \) and \( g_{E_T}(\zeta, \infty) \) be the Green functions of the domain \( \overline{\mathbb{C}} \setminus E_T \) with poles at the points 0 and \( \infty \), respectively. The regularity of the set \( E \) (and \( E_T \) correspondingly) implies the continuity of \( g_{E_T}(\zeta, 0) \) and \( g_{E_T}(\zeta, \infty) \) at all points different from 0 and \( \infty \), as well as the fact that these functions vanish at the points of \( E_T \). Therefore, for the \( \delta_1 > 0 \) there is a \( d_1 > 0 \), such that if \( t \in \mathbb{R} \) and \( \text{dist}(t, E) \leq d_1 \), then

\[
g_{E_T}(e^{it}, 0) < \frac{\delta_1^2}{2}. \tag{46}
\]
We choose $m$ sufficiently so large that for the set $E_{m}^{+}$ the condition $\text{dist}(t, E) \leq d_1$ for all $t \in E_{m}^{+}$ is satisfied. If $t \in E$ then

$$|T_n(t)Q_{nc}(t)| \leq \|T_n\|_E.$$  

If we write

$$T_n(t) = \sum_{j=0}^{n} (A_j \cos jt + B_j \sin jt)$$

$$= \sum_{j=0}^{n} (\Re A_j \cos jt + \Re B_j \sin jt) + i \sum_{j=0}^{n} (\Im A_j \cos jt + \Im B_j \sin jt),$$

we consider the algebraic polynomials

$$S_n^{(1)}(z) = \sum_{j=0}^{n} (\Re A_j - i\Re B_j) z^j,
S_n^{(2)}(z) = \sum_{j=0}^{n} (\Im A_j - i\Im B_j) z^j.$$  

It is easy to verify that $T_n(t) = F(e^{it})$ for all complex $t$, where

$$F(z) := \frac{1}{2} \left[ S_n^{(1)}(z) + \overline{S_n^{(1)} \left( \frac{1}{z} \right)} \right] + \frac{i}{2} \left[ S_n^{(2)}(z) + \overline{S_n^{(2)} \left( \frac{1}{z} \right)} \right]$$

is a rational function. We note that $\|F\|_{E_T} = \|T_n\|_E$ and apply an analog of the Bernstein-Walsh inequality (see e.g. [3], p. 64) to the rational function $F$ on $E_T$ and then use the fact that the domain $\mathbb{C} \setminus E_T$ is symmetric with respect to the unit circle. For simplicity, we put

$$g(z, w) = g_{\mathbb{C} \setminus E_T}(z, w)$$

for Green’s function of $E_T$. So, we have for $t \in \mathbb{R}$ that

$$|T_n(t)| = |F(e^{it})| \leq \|F\|_{E_T} \exp \left( n \left( g(e^{it}, 0) + g(e^{it}, \infty) \right) \right) = \|T_n\|_E \exp \left( 2n g(e^{it}, 0) \right).$$

Now if $t \in E_{m}^{+} \setminus E$ then it follows from (21) and (46) that

$$|T_n(t)Q_{nc}(t)| \leq \|T_n\|_E \exp \left( 2n g(e^{it}, 0) \right) \exp \left( -n\delta_1 \right)$$

$$\leq \|T_n\|_E \exp \left( n\delta_1^2 - n\delta_1 \right) \leq \|T_n\|_E$$

for sufficiently large $n$, and hence $\|T_nQ_{nc}\|_{E_{m}^{+}} \leq \|T\|_E$.

For $t \in [a - \rho, a]$ we have

$$\left| (T_nQ_{nc})^{(k)}(t) \right| \geq \left| T^{(k)}(t)Q_{nc}(t) \right| - \sum_{j=1}^{k} \binom{k}{j} \left| T_n^{(k-j)}(t)Q_{nc}^{(j)}(t) \right|.$$
Here \(1 - e^{-n\delta_1} \leq Q_{ne}(t) \leq 1\) and by \([26]\)
\[
\|Q_{ne}^{(j)}\|_E \leq C(n\varepsilon)^{2j}, \quad \|T_n^{(j)}\|_E \leq CN^{2j}\|T_n\|_E
\]

with some constant \(C\) for all \(j = 1, 2, \ldots, k\). Hence, if \(t \in [a - \rho, a]\) we get from the previous step applied to the trigonometric polynomial \(T_n(t)Q_{ne}(t)\) on the set \(E_m^+\) (which consists of finitely many intervals) that

\[
\left| T_n^{(k)}(t) \right| (1 - e^{-n\delta_1}) \leq \left| (T_nQ_{ne})^{(k)}(t) \right| + \sum_{j=1}^{k} \binom{k}{j} C^2||E||_E n^{2(k-j)}(ne)^{2j}
\leq (1+o(1))8^k \pi^{2k} \left( \frac{1}{2k-1} \right)! \left( \frac{1}{2k+1} \right)! \Omega \left( (E_m^+)_{\pi, e^{ia}} \right)^{2k} (n(1+\varepsilon))^{2k} \|T_nQ_{ne}\|_{E_m^+} + C_1 e^{2} n^{2k} \|T_n\|_E
\leq \frac{n^{2k}}{(2k-1)!} \|T_n\|_E \left( 1 + o(1) \right) (1 + \varepsilon)^{2k} \pi^{2k} \Omega \left( (E_m^+)_{\pi, e^{ia}} \right)^{2k} + C_1 e^{2} \|T_n\|_E.
\]

Since \(\varepsilon > 0\) and \(m\) are arbitrary, the inequality \([28]\) follows from Lemma \([1]\).

Fifth step. The regularity condition can be removed using the sets \(E_m^+\) and \(E_m^-\) from Ancona’s theorem (interval condition \([2]\) implies \([a - \rho, a] \subset E\), hence \(\text{cap}(E) > 0\)). Indeed,

\[
\|T_n^{(k)}\|_{[a-\rho,a]} \leq (1 + o_m(1)) \frac{n^{2k}}{(2k-1)!} \pi^{2k} \Omega \left( (E_m^-)_{\pi, e^{ia}} \right)^{2k} \|T_n\|_{E_m^-}
\leq (1 + o_m(1)) \frac{n^{2k}}{(2k-1)!} \pi^{2k} \Omega \left( (E_m^-)_{\pi, e^{ia}} \right)^{2k} \|T_n\|_E
\]

where \(o_m(1)\) depends on \(E_m^-\) too.

It follows from Lemma \([1]\) that \(\Omega( (E_m^-)_{\pi, e^{ia}} \) can be made arbitrary close to \(\Omega( (E)_{\pi, e^{ia}} \) by choosing \(m\) large enough. Hence the inequality \([28]\) holds in this case too.

Now we investigate the sharpness, that is, we are going to establish \([29]\). As above, first we show it for the case when \(E\) is a union of finitely many intervals. We select a \(T\)-set as in Section \([2,1]\) for which \(\Omega( (E)_{\pi, e^{ia}} \) is close to \(\Omega( (E)_{\pi, e^{ia}}\), say \(\Omega( (E')_{\pi, e^{ia}} \geq \Omega( (E)_{\pi, e^{ia}})(1 - \varepsilon)\) for some given \(\varepsilon > 0\).

By \([32]\)
\[
|U'_N(a)| = 8\pi^2 N^2 \Omega(E', e^{ia})^2.
\]

(47)

Now note that if \(T_i(x) = \cos(i\arccos(x))\) are classical Chebyshev polynomials, then \(T_i(t) := T_i(U_N(t))\) is a trigonometric polynomial of degree \(lN\) for which

\[
E' = \{ x | T_i(U_N(t)) \in [-1, 1] \}.
\]

Since
\[
|T_i^k(+1)| = \frac{l^2(l^2 - 1) \ldots (l^2 - (k-1)^2)}{(2k-1)!} =: C_{l,k}.
\]
and \(| \mathbf{T}^{(k)}_n(a) | = (\mathbf{T}_l(U_N))^{(k)}(a) | = (1 \pm o(1)) C_{l,k} N^{2k} r^{2\pi k} \Omega(E, e^{i\alpha})^{2k} \),

and here, in view of (31),

\[ C_{l,k} N^{2k} \Omega(E, e^{i\alpha})^{2k} \geq (1 - o(1)) \frac{I^{2k}}{(2k - 1)!} \Omega^{2k} \Omega(E, e^{i\alpha})^{2k} (1 - \varepsilon)^{2k}. \]

Since \( E \subset E' \), we have
\[ \|T_n\|_E \leq \|T_n\|_{E'} = \|T_r\|_{[-1,1]} = 1, \]

and so from \( n = lN \) we get
\[ \|T_n^{(k)}(a)\| \geq (1 - o(1))^2 (1 - \varepsilon)^{2k} \frac{n^{2k}}{(2k - 1)!} \Omega^{2k} \Omega(E, e^{i\alpha})^{2k} \|T_n\|_E. \]

This is only for integers \( n \) of the form \( n = lN \). For others just use \( T_n(t) = T_{l[n/N]}(U_N(t)) + \delta \cos(nt) \) with \( \delta > 0 \) very small. Since here \( \varepsilon = \varepsilon_N > 0 \) is arbitrary, \([29] \) follows if we let \( N \) tend to \( \infty \) slowly and at the same time \( U_N^{-1}[-1,1] \) approaches \( E \), as \( n \to \infty \) (in which case we have \( \varepsilon_N \to 0 \)).

In the general case we consider the sets \( E_m^+ \) that are unions of finitely many intervals. Hence, we may use the last result for \( E_m^+ \), namely, there is a sequence of nonzero trigonometric polynomials \( \{T_{m,n}\}_{n=1}^\infty \), \( \text{deg}(T_{m,n}) \leq n \), such that
\[ \|T_{m,n}(a)\| \geq (1 - o_{E_\infty}(1)) n^{2k} \Omega \left( \left( E_m^+ \right)^{2k} \Omega e^{i\alpha} \right) \frac{8^k \pi^{2k}}{(2k - 1)!} \|T_{m,n}\|_{E_m^+}, \]

where \( o_{E_\infty}(1) \) depends on \( E_m^+ \), and it tends to 0 as \( n \to \infty \) for any fixed \( m \). Since \( E \subset E_m^+ \), we have \( \|T_{m,n}\|_{E_m^+} \geq \|T_{m,n}\|_E \) and hence
\[ \|T_{m,n}^{(k)}(a)\| \geq (1 - o_{E_\infty}(1)) n^{2k} \Omega \left( \left( E_m^+ \right)^{2k} \Omega e^{i\alpha} \right) \frac{8^k \pi^{2k}}{(2k - 1)!} \|T_{m,n}\|_E. \]

By Lemma \([1]\) and choosing \( m \) sufficiently large, \( \Omega \left( \left( E_m^+ \right)^{2k} \Omega e^{i\alpha} \right) \) can be made arbitrarily close to \( \Omega(E, e^{i\alpha}) \). Therefore, \([29] \) follows for \( T_n := T_{m,n} \) if \( m_n \) goes slowly to infinity as \( n \to \infty \).

Now if \( H \) denotes the shorter arc on \( \mathbb{T} \) connecting the points \( e^{i(a - \rho)} \) and \( e^{i\alpha} \) then we have the following assertion.

**Corollary 7.** Under the conditions mentioned above for any algebraic polynomial \( P_n \), with degree \( n \), we have
\[ \|P_n^{(k)}\|_H \leq (1 + o(1)) n^{2k} \Omega(E, e^{i\alpha})^{2k} \frac{2^k \pi^{2k}}{(2k - 1)!} \|P_n\|_E. \] (48)
This inequality is sharp, for there is a sequence of polynomials \( P_n \neq 0 \), \( n = 1, 2, \ldots \), such that

\[
\left| P_n^{(k)}(e^{ia}) \right| \geq (1 - o(1))n^{2k} \Omega(E, \Omega E, e^{ia})2^{k} \approx \frac{2k}{(2k - 1)!} \| P_n \|_{E_T}. \tag{49}
\]

The quantity \( o(1) \) depends on \( E \) and \( k \) and tends to 0 as \( n \to \infty \).

**Proof.** We may assume that \( n \) is even (because \((n+1)^2/n^2 = 1 + o(1))\). We consider the trigonometric polynomial \( T_n/2(t) = e^{-itn/2}P_n(e^{it}) \). So, (48) follows now from applying Theorem 6 to \( T_n/2 \).

Concerning (49), existence of such polynomials, in view of the remark above, follows from existence of trigonometric polynomials \( T_n \) for which (29) holds.

\[ \Box \]

4 Higher order Bernstein-type inequalities and their sharpness

Let \( E \subset (-\pi, \pi) \) be a compact subset, and fix a point \( z_0 = e^{it_0} \) which is in the one dimensional interior of \( E_T \). That is, \( \{ \exp(it) : t_0 - \delta < t < t_0 + \delta \} \subset E_T \) for some small \( \delta > 0 \). Denote by \( \partial/\partial n_+ \) and \( \partial/\partial n_- \) the outward and inward normal derivatives (w.r.t. the unit circle) correspondingly. Then (see [17], formulas (23) and (24) on p. 349)

\[
\frac{1}{2} \left( 1 + 2\pi \omega_{E_T}(e^{it}) \right) \cdot \frac{\partial g(e^{it}, \infty)}{\partial n_+} = \max \left( \frac{\partial g(e^{it}, \infty)}{\partial n_+}, \frac{\partial g(e^{it}, \infty)}{\partial n_-} \right)
\]

where \( g(z, w) = \partial_{C \setminus E_T}(z, w) \) is Green’s function of \( C \setminus E_T \) and \( \omega_{E_T}(\cdot) \) denotes the density of the equilibrium measure (w.r.t. arc length on the unit circle).

Now let us consider higher order Bernstein-type inequalities for trigonometric polynomials.

**Theorem 8.** Let \( E \subset (-\pi, \pi) \) be a compact set and \( k \) be a positive integer. Fix a closed interval \( E_0 \subset \text{Int} E \) (subset of the one dimensional interior of \( E \)). Then there exists \( C = C(E, E_0, k) > 0 \) such that for all trigonometric polynomial \( T_n \) with degree \( n \), we have for \( t \in E_0 \)

\[
\left| T_n^{(k)}(t) \right| \leq (1 + o(1))n^k \left( 2\pi \omega_{E_0}(e^{it}) \right)^k \| T_n \|_{E_T}. \tag{50}
\]

where \( o(1) \) is uniform in \( t \in E_0 \) and uniform among all trigonometric polynomials having degree at most \( n \) and tends to 0 as \( n \to \infty \).

**Proof.** We prove the theorem by induction on \( k \), the case \( k = 1 \) was done in [13, Theorem 4].

Let \( V(t) = 2\pi \omega_{E_0}(e^{it}) \).
Select a closed set $E_0^* \supset E_0$ such that $E_0^*$ has no common endpoints either with $E_0$ or with $E$.

Consider any $\delta > 0$ such that the intersection of $E$ with the $\delta$-neighborhood of $E_0$ is still subset of of $E_0^*$, and set $f_{k,n,t_0}(t) := T_n^{(k)}(t)Q(t)$, where $Q(t) = Q_{n,t_0}(t)$ is a fast decreasing trigonometric polynomial from Theorem 3 for $t_0 \in E_0$ ($\alpha'$ and $\beta'$ from Theorem 3 are chosen such a way that the interval $[\alpha', \beta']$ is in the $\delta$-neighborhood of $E_0$).

By (23) and (26), for this $f_{k,n,t_0}$ we have the upper bound

$$O(n^{2k}) \exp \left(-\delta_1 n^{1/3}\right) \|T_n\|_E = o(1)\|T_n\|_E$$

on $E$ outside the $\delta$-neighborhood of $t_0$ with $\delta_1 > 0$ (uniform in $t_0 \in E_0$).

In the $\delta$-neighborhood of any $t_0 \in E_0$, by $\|Q\|_E \leq 1$ and by induction hypothesis applied to $T_n$ and to $E_0^*$, we have

$$|f_{k,n,t_0}(t)| \leq (1 + o(1))n^k\|T_n\|_E V(t)^k \leq (1 + o(1))n^k(1 + \varepsilon)^k\|T_n\|_E V(t_0)^k;$$

where $\varepsilon \to 0$ as $\delta \to 0$. Here we used that by the continuity of $V(t)$, if $t_0 \in E_0$ and $|t - t_0| < \delta$, then $V(t) \leq (1 + \varepsilon)V(t_0)$ with some $\varepsilon$ that tends to 0 as $\delta \to 0$. Therefore, $f_{k,n,t_0}(t)$ is a trigonometric polynomial in $t$ of degree at most $n + n^{1/3}$ for which

$$\|f_{k,n,t_0}\| \leq (1 + o(1))n^k\|T_n\|_E V(t_0)^k.$$

Upon applying Lukashov’s theorem from [13, Theorem 4] to the trigonometric polynomial $f_{k,n,t_0}(t)$ we obtain

$$|f_{k,n,t_0}(t_0)| \leq (1 + o(1))n^{k+1}\|T_n\|_E V(t_0)^{k+1}. \quad (51)$$

Since (recall that $Q(t_0) = 1$)

$$f'_{k,n,t_0}(t_0) = T_n^{(k+1)}(t_0) + T_n^{(k)}(t_0)(Q(t_0))',$$

and the second term on the right is at most $O(n^k)O(n^{2/3})\|T_n\|_E$ in modulus, by (26) and by the induction assumption, from (51) we get (50). It follows from the proof that the estimate is uniform in $t_0 \in E_0$.

\[\square\]

**Corollary 9.** Let $E \subset (-\pi, \pi)$ be again a compact set and $k$ be a positive integer. Fix a closed interval $E_0 \subset \text{Int} E$. Then there exists $C = C(E, E_0, k) > 0$ such that for all algebraic polynomial $P_n$ with degree $n$, we have for $z = e^{it}, \ t \in E_0$

$$|P_n^{(k)}(z)| \leq (1 + o(1)) \frac{n^k}{2^k} (1 + 2\pi \omega_{E_0}(z)) \|P_n\|_{E_0^*} \quad (52)$$

where $o(1)$ is uniform in $z = e^{it}, \ t \in E_0$ and independent of $P_n$, but it tends to 0 as $n \to \infty$. 

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Proof. As in the proof of Corollary 7, we may assume that \( n \) is even (because \( (n + 1)^2/n^2 = 1 + o(1) \)) and consider the trigonometric polynomial \( T_{n/2}(t) = e^{-itn/2}P_n(e^{it}) \). By Theorem 8 we get

\[
(1 + o(1)) \frac{n^k}{2^k} \left( 2\pi \omega_{E_T}(e^{it}) \right)^k \|T_{n/2}\|_E \geq |T_n^{(k)}(t)|
\]

\[
\geq \left| (P_n(e^{it}))^{(k)} \right| - \sum_{j=0}^{k-1} \binom{k}{j} \left( P(e^{it}) \right)^{(j)} (e^{-itn/2})^{(k-j)} \left| T_k^{(j)}(t) \right|.
\]

It, together with Faà di Bruno’s formula (1) and Theorem 8 yields that

\[
\left| P_n^{(k)}(z) \right| \leq (1 + o(1)) \frac{n^k}{2^k} \left( 2\pi \omega_{E_T}(z) \right)^k \left( 1 + 2\pi \omega_{E_T}(z) \right)^k \|P_n\|_{E_T}
\]

\[
\leq (1 + o(1)) \frac{n^k}{2^k} (1 + 2\pi \omega_{E_T}(z))^k \|P_n\|_{E_T}.
\]

Corollary 9 extends Theorem 1 of the paper [17] to higher derivatives of algebraic polynomials and the proof of sharpness is similar to the proof of [17], Theorem 2.

\begin{proof}

We enclose \( E_T \) into a set \( G \) with the following properties:

- \( G \) is a finite union of disjoint \( C^2 \) smooth Jordan domains: there are finitely many disjoint \( C^2 \) Jordan curves \( S_1, \ldots, S_m \) such that if \( G_j \) is the bounded connected components of \( C \setminus S_j \), then \( G = \bigcup_{j=1}^{m} G_j \).
- \( E_T \) is a boundary arc of the boundary \( \partial G \).
- the component of \( G \) that contains \( z \) lies in the closed unit disk,
- every point of \( G \) is of distance \( \leq \eta \) from a point of \( E_T \), where \( \eta \) is a given positive number.

Then the boundary \( \Gamma = \partial G = \bigcup_{j=1}^{m} S_j \) is a family of disjoint Jordan curves. Furthermore, let \( n_+ = z \) be the normal at \( z \) to \( \Gamma \) pointed to the interior of \( \Omega = \overline{C} \setminus G \).

\end{proof}
If $\varepsilon > 0$ is given, then for sufficiently small $\eta$ we have (see e.g. [15], pp. 350-351)

$$\frac{\partial \varphi_\Omega(z, \infty)}{\partial n_+} \geq (1 - \varepsilon) \frac{\partial \varphi_\mathrm{ET}(z, \infty)}{\partial n_+}.$$  \hspace{1cm} (53)

By the sharp form of the Hilbert lemniscate theorem [15], Theorem 1.2, there is a Jordan curve $\sigma$ such that

- $\sigma$ contains $\Gamma$ in its interior except for the point $z$, where the two curves touch each other,
- $\sigma$ is a lemniscate, i.e. $\sigma = \{ \zeta : |V_N(\zeta)| = 1 \}$ for some algebraic polynomial $V_N$ of degree $N$, and
- $\frac{\partial \varphi_{\mathrm{ET}}(z, \infty)}{\partial n_+} \geq (1 - \varepsilon) \frac{\partial \varphi_\Omega(z, \infty)}{\partial n_+}$. \hspace{1cm} (54)

We may assume that $V_N'(z) > 0$. The Green’s function of the outer domain of $\sigma$ is $\frac{1}{N} \log |V_N(\cdot)|$, and its normal derivative is

$$\frac{\partial \varphi_{\mathrm{ET}}(z, \infty)}{\partial n_+} = \frac{1}{N} |V_N'(z)| = \frac{1}{N} V_N'(z).$$

Consider now, for all large $n$, the polynomials $P_n(.) = V_N(.)^{[n/N]}$. This is a polynomial of degree at most $n$, its supremum norm on $\sigma$ is 1, and by Faà di Bruno formula (1), it can be shown that (see also [8], subsection 10.2)

$$\left| P_n^{(k)}(z) \right| = n^k \left( \frac{\partial \varphi_{\mathrm{ET}}(z, \infty)}{\partial n_+} \right)^k + O(n^{k-1}).$$

Thus, in view of (53) and (54), we may continue

$$\left| P_n^{(k)}(z) \right| \geq (1 - \varepsilon)^{2k} n^k \left( \frac{\partial \varphi_{\mathrm{ET}}(z, \infty)}{\partial n_+} \right)^k + O(n^{k-1}).$$

Note also that $\|P_n\|_E \leq \|P_n\|_\sigma = 1$ by the maximum principle.

**Corollary 11.** Under assumption of Theorem 8, inequality (50) is sharp, for there is a sequence of trigonometric polynomials $T_n \neq 0$, $n = 1, 2, \ldots$, such that

$$\left| T_n^{(k)}(t) \right| \geq (1 - o(1))n^k (2\pi \omega_E \langle e^it \rangle)^k \|T_n\|_E .$$

where $o(1)$ depends on $E$ and $k$ and tends to 0 as $n \to \infty$. \hspace{1cm} \Box

**Proof.** Existence of such trigonometric polynomials $T_n$ follows immediately from the existence of corresponding (in the sense of the proof of Corollary 9) algebraic polynomials $P_{2n}$ from Corollary 9. \hspace{1cm} \Box
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References


Sergei Kalmykov
School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dongchuan RD, Shanghai, 200240, P.R. China
Far Eastern Federal University, 8 Sukhanova Street, Vladivostok, 690950, Russia
email address: sergeykalmykov@inbox.ru

Béla Nagy
MTA-SZTE Analysis and Stochastics Research Group, Bolyai Institute, University of Szeged, Szeged, Aradi v. tere 1, 6720, Hungary
email address: nbela@math.u-szeged.hu