Higher Markov and Bernstein inequalities and fast decreasing polynomials with prescribed zeros

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Abstract

Higher order Bernstein- and Markov-type inequalities are established for trigonometric polynomials on compact subsets of the real line and algebraic polynomials on compact subsets of the unit circle. In the case of Markov-type inequalities we assume that the compact set satisfies an interval condition.

Keywords: trigonometric polynomials, algebraic polynomials, Bernstein-type inequalities, equilibrium measure, Green's function, fast decreasing polynomials.

Classification (MSC 2010): 41A17, 30C10

1 Introduction

Two of the most classical polynomial inequalities are the Bernstein inequality (see [2], p. 233 Theorem 5.1.7 or [14], p. 532, Theorem 1.2.5)

$$|P_n'(x)| \le \frac{n}{\sqrt{1-x^2}} \|P_n\|_{[-1,1]}, \quad x \in (-1,1),$$

and the Markov inequality (see [2], p. 233 Theorem 5.1.8 or [14], p. 529 Theorem 1.2.1)

$$\|P_n'\|_{[-1,1]} \le n^2 \|P_n\|_{[-1,1]},$$

where P_n is an algebraic polynomial of degree of at most n, and $\|\cdot\|_X$ denotes the sup-norm over the set X. For a trigonometric polynomial T_n of the degree at most n the following Bernstein-type inequality holds (established by M. Riesz, see [14], p. 532 Theorem 1.2.4 or [2], p. 232 Theorem 5.1.4)

$$\|T_n'\|_{[0,2\pi]} \le n \|T_n\|_{[0,2\pi]}.$$

There is also an analogue of this inequality for trigonometric polynomials on an interval less than the period see [2] p. 243. In 2001, Totik developed the method of polynomial inverse images to prove an asymptotically sharp Bernstein- and Markov-type inequalities for algebraic polynomials on several intervals [25], and in [28] asymptotically sharp inequalities were also obtained for trigonometric polynomials on several intervals and for algebraic polynomials on several circular arcs on the complex plane. The case of one circular arc was considered earlier in [16]. In recently published paper [7] algebraic polynomials on sets satisfying (2) were considered, for trigonometric polynomials, see [6]. The next step in generalization of these result was done in [23], where asymptotic higher order Markov-type inequalities for algebraic polynomials on compact sets satisfying (2) were established.

The purpose of the present paper is to extend these results to trigonometric polynomials and to algebraic polynomials on subsets of the unit circle and to present a new type of fast decreasing polynomials. Briefly, the approach of Totik-Zhou [23] was to establish the Markov-type inequality for T-sets, then for general sets and use Faà di Bruno's formula and Remez inequality near interior critical points. The difference here is that we developed fast decreasing polynomials with prescribed zeros to deal with interior critical points. Moreover, we also establish Bernstein-type inequality.

Sharp higher order Markov-type inequality is established for sets satisfying the interval condition (2). At interior points sharp Bernstein-type inequality is also derived which involves much slower growth order ($O(n^{2k})$ at endpoints vs. $O(n^k)$ at interior points where k-th derivatives are considered).

The structure of the paper is the following. First, notation is introduced, and some known, basic results about T-sets are mentioned. Then the important density results (for T-sets and regular sets) are recalled. New results are in Section 3. A construction of fast decreasing polynomials with prescribed zeros can also be found here. A preliminary, "rough" Markov- and Bernstein-type inequalities are needed for special sets. Then asymptotically sharp Markov-type inequality is formulated for higher derivatives of trigonometric polynomials and for algebraic polynomials on subsets of the unit circle. Finally, asymptotically sharp Bernstein-type inequalities are established in the trigonometric case as well as in the algebraic case.

2 Notation, background

We denote by $\mathbf{R}$ the real line, by $\mathbf{C}$ the complex plane, by $\overline{\mathbf{C}}$ the extended complex plane, and by $\mathbb{T}$ the unit circle and by $\mathbf{N}$ the nonnegative integers.

We use Faà di Bruno's formula (or Arbogast's formula; see [9], p. 17 or [21], pp. 35-37 or [5]): if f and g are k times differentiable functions, then

$$\frac{d^k}{dx^k} f(g(x)) = \sum \frac{k!}{m_1! m_2! \ldots m_k!} f^{(m_1+m_2+\ldots+m_k)}(g(x)) \prod_{j=1}^{k} \left(\frac{g^{(j)}(x)}{j!} \right)^{m_j} \quad (1)$$

where the summation is for all nonnegative integers $m_1, m_2, \ldots, m_k$ such that

$$1m_1 + 2m_2 + \ldots + km_k = k.$$

Let $E \subset [-\pi, \pi)$ be a set which is closed in $[-\pi, \pi)$. Since we do not consider $E = [-\pi, \pi)$ (it is classical), we may assume that $E \subset (-\pi, \pi)$. We consider the corresponding set on the unit circle

$$E_{\mathbb{T}} := \{\exp(it) : \ t \in E\}.$$

We use the interval condition: a compact set $E \subset (-\pi, \pi)$ satisfies the interval condition at $a \in E$ if there is a $\rho > 0$ such that

$$[a - 2\rho, a] \subset E \text{ and } (a, a + 2\rho) \cap E = \emptyset. \tag{2}$$

We use potential theory, for a background, we refer to [20] or [22]. For a compact set $K \subset \mathbf{C}$, its capacity is denoted by $\mathrm{cap}(K)$. If $\mathrm{cap}(K) > 0$, then the equilibrium measure is denoted by ν_K. It is known that if $K \subset \mathbf{R}$ is a compact set ν_K is absolutely continuous with respect to Lebesgue measure at interior points of K and its density is denoted by $\omega_K(t)$. It is also known that if $E \subset (-\pi, \pi)$ satisfies the interval condition at a point $a \in E$, then $\sqrt{|t-a|}\omega_E(t)$ has a finite, positive limit as $t \to a$. Similarly, we say that the compact set $K \subset \mathbb{T}$ satisfies the interval condition at e^{ia} where $a \in (-\pi, \pi)$ if $K = E_{\mathbb{T}}$ and for some E, E satisfies the interval condition at a. Furthermore, if K satisfies the interval condition at e^{ia} ($a \in (-\pi, \pi)$), then $\sqrt{|e^{it} - e^{ia}|}\omega_K(e^{it})$ has a finite, positive limit as $t \to a$ too. Hence we introduce

$$\Omega(E, a) := \lim_{t \to a} \sqrt{|t-a|}\, \omega_E(t),$$
$$\Omega(K, e^{ia}) := \lim_{t \to a} \sqrt{|e^{it} - e^{ia}|}\, \omega_K(e^{it}).$$

It is worth noting that $\Omega(.,.)$ is monotone with respect to the set, that is, if $E_1 \subset E_2 \subset [-\pi, \pi)$, and both satisfy the interval condition at a, then $\Omega(E_2, a) \leq \Omega(E_1, a)$. Similar assertion holds for the unit circle.

In the finitely many arcs case, there is a very useful representation of the density of the equilibrium measure (see [19], Lemma 4.1 and also formula (5.11)): let $K = \cup_{j=1}^m \{\exp(it) : a_{2j-1} \leq t \leq a_{2j}\}$ where $-\pi < a_1 < a_2 < \ldots < a_{2m-1} < a_{2m} < \pi$ and put $a_{2m+1} := 2\pi + a_1$. Then there exist $\tau_j \in (a_{2j}, a_{2j+1})$, $j = 1, \ldots, m$ such that

$$\int_{a_{2j}}^{a_{2j+1}} \frac{\prod_{j=1}^m (e^{it} - e^{i\tau_j})}{\sqrt{\prod_{j=1}^m (e^{it} - e^{ia_{2j-1}})(e^{it} - e^{ia_{2j}})}} dt = 0 \tag{3}$$

where, to be definite, the branch of the square root is chosen so that $\sqrt{z} \to \infty$ as $z \in \mathbf{R}$, $z \to +\infty$. Actually it should hold that

$$(-1)^m i \prod_j e^{i\tau_j} = \sqrt{\prod_j e^{i(a_{2j-1} + a_{2j})}}$$

but actually the other branch would be just as fine, since the right hand side in (3) is 0. Then

$$\omega(K, e^{it}) = \frac{1}{2\pi} \frac{\prod_{j=1}^m |e^{it} - e^{i\tau_j}|}{\sqrt{\prod_{j=1}^m |e^{it} - e^{ia_{2j-1}}|\, |e^{it} - e^{ia_{2j}}|}}, \quad t \in \mathrm{Int}\, K$$

see [19], formula (5.11). In this case,

$$\Omega\left(K, e^{ia_k}\right) = \frac{1}{2\pi}\frac{\prod_{j=1}^{m}\left|e^{ia_k} - e^{i\tau_j}\right|}{\sqrt{\prod_{j=1,\dots,2m, j\neq k}\left|e^{ia_k} - e^{ia_j}\right|}}.$$

2.1 Density results

We use special sets on $(-\pi,\pi)$. A set $E\subset(-\pi,\pi)$ is called T-set, if

$$E = \{t\in(-\pi,\pi) : |U_N(t)|\le 1\} \tag{4}$$

for some (real) trigonometric polynomial U_N with degree N which attains $+1$ and -1 $2N$-times. For a background on T-sets, we refer to Section 3 in [28].

We define

$$M(E,a_j) = M_{a_j} := \frac{\prod_{l=1}^{m}\left|e^{ia_j} - e^{i\tau_l}\right|^2}{\prod_{l=1,\dots,2m, l\neq j}\left|e^{ia_j} - e^{ia_l}\right|}$$

and obviously,

$$M(E,a_j) = 4\pi^2\Omega^2(E_{\mathbb{T}}, e^{ia_j}).$$

Now we recall some monotonicity and continuity results regarding $\Omega(E,a)$ and $M(E,a)$.

For any $\varepsilon>0$, by Lemma 3.4 from [28] (see p. 3001) we can choose an admissible polynomial U_N such that the inverse image set $E' = (U_N^{-1}[-1,1])\cap[-\pi,\pi] = \cup_{j=1}^{m}[a'_{2j-1}, a'_{2j}]$ consists of m intervals and it lies close to E, that is $|a'_j - a_j| < \varepsilon$ for all $j=1,\dots,2m$ and $E'\subset E$. Also we may assume that $a\in E'$. Again j_0 is such that $a\in[a'_{2j_0-1}, a'_{2j_0}]$ and actually $a = a'_{2j_0}$. For numbers τ_i in (3) it is clear that they are C^1-functions of the endpoints a_j. Then with $M'_a := M(E',a)$, we have $\lim_{\varepsilon\to 0} M'_a = M_a$. By the monotonicity of $\Omega(.,.)$ in the first variable, we immediately have that $M_a\le M'_a$.

In other words, for any $\varepsilon>0$, there exists a T-set $E'\subset E$, $a\in E'$ such that $\Omega^2(E'_{\mathbb{T}}, e^{ia}) \le (1+\varepsilon)\Omega^2(E_{\mathbb{T}}, e^{ia})$.

Consider an arbitrary compact set $E\subset(-\pi,\pi)$ satisfying the interval condition (2), and assume that E is not a union of finitely many intervals. The set $[-\pi,\pi]\setminus E$ consists of finitely or countably many intervals open in $[-\pi,\pi]$:

$$[-\pi,\pi]\setminus E = \bigcup_{j=0}^{\infty} I_j$$

To be definite, we assume that I_0 contains $(a, a+2\rho)$. Further, for $m\ge 0$ we consider the set

$$E_m^+ = [-\pi,\pi]\setminus\left(\bigcup_{j=0}^{m} I_j\right) = \bigcup_{j=1}^{m'}[a_{j,m'}, b_{j.m'}],$$

$$a_{1,m'}\le b_{1,m'} < a_{2,m'}\le b_{2,m'} < \dots < a_{m',m'}\le b_{m',m'} = b_{0,m'}$$

where $m' = m + 1$ (note here, by our assumption $E \subset (-\pi, \pi)$).

Obviously, E_m^+ contains E and satisfies the interval condition (2). If $a_{j,m'} = b_{j,m'}$ for some j, then we replace this degenerated interval by the interval

$$[a_{j,m'} - \lambda_m, a_{j,m'} + \lambda_m] \bigcap [-\pi, \pi],$$

where $\lambda_m < 1/m$ is chosen to be so small that the interval condition (2) is still satisfied. For the set obtained this way we preserve the notation E_m^+.

We also use the famous result of Ancona (see [1]). If $K \subset \mathbb{T}$ is any compact set, $\mathrm{cap}(K) > 0$, then for any $\varepsilon > 0$ there exists $K_1 \subset K$ compact set which is regular for the Dirichlet problem and $\mathrm{cap}(K) \le \mathrm{cap}(K_1) + \varepsilon$. Furhermore, it is easy to see that if K satisfies the interval condition (2), then K_1 can be chosen such that it satisfies (2) too. Let E_m^- be the set coming from Ancona's theorem applied to $E_{\mathbb{T}}$ with $\varepsilon = 1/m$ and also satisfying the interval condition (2).

Lemma 1. *For the two sets E_m^+ and E_m^- introduced above, we have $\Omega\left((E_m^{\pm})_{\mathbb{T}}, e^{ia}\right) \to \Omega(E_{\mathbb{T}}, e^{ia})$ holds true as $m \to \infty$.*

For a proof, see e.g. [7], p. 1295, Proposition 2.3.

3 New results

We need fast deceasing polynomials with prescribed zeros and rough Markov- and Bernstein-type inequalities.

3.1 Fast decreasing trigonometric and algebraic polynomials with prescribed zeros

Special fast decreasing polynomials with prescribed zeros are constructed in this subsection. First, their existence are established on the real line, then in the trigonometric case.

We tried to find this type of fast decreasing polynomials in the existing literature (e.g. in [12], [4], [24], [26],[27],[29], [10] and Lemma 4.5 on p. 3012 in [28]), but we did not find the following two results. Further, possible applications may include estimates for Christoffel functions, etc.

Theorem 2. *Let $a_0 < a_1 < \ldots < a_{l_0} < a' < a < x_0 < b < b' < a_{l_0+1} < \ldots < a_l < a_{l+1}$ be fixed and $k_0, k_1, \ldots, k_l$ be positive integers. Put $Z(x) := \prod_{j=1}^{l} (x - a_j)^{k_j}$. Then there exists $\delta_1 > 0$ such that for all large m there exists* *a polynomial $Q(x)$ with degree at most m such that*

$$Q(x_0) = 1, \tag{5}$$

$$Q^{(j)}(x_0) = 0, \quad j = 1, \ldots, k_0, \tag{6}$$

$$|Q(x)| < 1 \text{ if } x \in [a_0, a_{l+1}], x \neq x_0, \tag{7}$$

$$|Q(x) - 1| \leq \exp(-\delta_1 m) \text{ for } x \in [a, b], \tag{8}$$

$$|Q(x)| \leq \min(1, |Z(x)|) \exp(-\delta_1 m) \text{ for } x \in [a_0, a'] \cup [b', a_{l+1}], \tag{9}$$

$$Q(x) \text{ is strictly monotone on } [a', a] \text{ and on } [b, b'], \tag{10}$$

$$Q^{(k)}(a_j) = 0, \quad j = 1, \ldots, l, k = 0, 1, \ldots, k_j, \tag{11}$$

$$Q(x) \geq 0 \text{ for } x \in [a_0, a_{l+1}]. \tag{12}$$

Proof. In this proof several new pieces of notation are introduced which are used here only and constants are not redefined from line to line in this proof just for sake of convenience.

Consider S, which will be a polynomial satisfying all but one properties, in the form

$$S(x) = C_1 \int_{a_1}^{x} Z_1(t)\, P_1(t)\, R(t)(t - x_0)^{k_0'}\, dt \tag{13}$$

where

$$Z_1(t) := \prod_{j=1}^{l} (t - a_j)^{k_j'}, \quad R(\underline{\tau}; t) = R(t) := \prod_{\substack{j=1 \\ j \neq l_0}}^{l-1} (t - \tau_j)$$

$$P_0(t) = P_0(\delta, \mu; t) := \left(1 - \left(\frac{x - \delta}{c_2}\right)^2\right)^{\mu}$$

$$P_1(\alpha, \beta, \lambda, \mu; t) = (1 - \lambda) P_0(\alpha, \mu; t) + \lambda P_0(\beta, \mu; t)$$

and where $k_0' = k_0$ if k_0 is odd and $k_0' = k_0 + 1$ if k_0 is even, and for $j = 1, \ldots, l$, $k_j' = k_j$ if k_j is even and $k_j' = k_j + 1$ if k_j is odd, and $\tau_j \in [a_j, a_{j+1}]$, $j = 1, \ldots, l-1$, $j \neq l_0$, and $a' < \alpha < a < b < \beta < b'$, $\alpha := (a + a')/2$, $\beta := (b + b')/2$ and μ is large positive integer and $c_2 := a_{l+1} - a_0$, $\lambda \in [0, 1]$. If some of the parameters are fixed or unimportant in the current consideration, then we leave them out, e.g. $P_0(t) = P_0(\delta, \mu; t)$ and $P_1(t) = P_1(\mu; t) = P_1(\lambda, \mu; t) = P_1(\alpha, \beta, \lambda, \mu; t)$.

The key observation is that if $S(a_j) = 0$ for some j, then we immediately have that $S^{(k)}(a_j) = 0$, $k = 0, 1, 2, \ldots, k_j$.

Some obvious properties immediately follow from the definitions: $Z_1(t) \geq 0$ (this is why we increased the "multiplicities"), $P_0(t), P_1(t) \geq 0$ too, $\max_{a_0 \leq t \leq a_{l+1}} P_1(t) \geq 1/2$. Furthermore, the degree of R is $l - 2$ and R has the same sign over (a', b'). For simplicity, denote $\underline{\tau}_1 := (\tau_1, \ldots, \tau_{l_0 - 1})$, $\underline{\tau}_2 := (\tau_{l_0+1}, \ldots, \tau_l)$ and (slightly abusing the notation) $\underline{\tau} := (\underline{\tau}_1, \underline{\tau}_2) = (\tau_1, \ldots, \tau_{l_0-1}, \tau_{l_0+1}, \ldots, \tau_l)$ and $(\underline{\tau}_1, \lambda, \underline{\tau}_2) := (\tau_1, \ldots, \tau_{l_0-1}, \lambda, \tau_{l_0+1}, \ldots, \tau_l)$. Finally, the degree of S is $k_1' + \ldots + k_l' + 2\mu + l - 2 + k_0' + 1 = 2\mu + const$.

Poincaré-Miranda theorem (see e.g. [11], p. 547 or [18], pp. 152-153) helps to find a solution so that S vanishes at all prescribed a_j's. In detail, put $\mathcal{R} := [a_1, a_2] \times \ldots \times [a_{l_0-1}, a_{l_0}] \times [0,1] \times [a_{l_0+1}, a_{l_0+2}] \times \ldots \times [a_{l-1}, a_l]$ and for $j = 1, \ldots, l$ let $f_j : \mathcal{R} \to \mathbf{R}$,

$$f_j(\underline{\tau}_1, \lambda, \underline{\tau}_2) := \int_{a_j}^{a_{j+1}} Z_1(t) P_1(\lambda, \mu; t)\, R(\underline{\tau}; t)(t - x_0)^{k_0'}\, dt.$$

Now we verify the signs of these functions on opposite sides of $\mathcal{R}$: if $j = 1, \ldots, l$, $j \neq l_0$, then $A_j := \{(\tau_1, \ldots, \tau_{l_0-1}, \lambda, \tau_{l_0+1}, \ldots, \tau_l) \in \mathcal{R} : \tau_j = a_j\}$ and $B_j := \{(\tau_1, \ldots, \tau_{l_0-1}, \lambda, \tau_{l_0+1}, \ldots, \tau_l) \in \mathcal{R} : \tau_j = a_{j+1}\}$ are the opposite sides. We have to treat the case $j < l_0$ and the case $j > l_0$ separately. If $(\underline{\tau}_1, \lambda, \underline{\tau}_2) \in A_j$, then $R(t)$ has the same sign all over (a_j, a_{j+1}) and $\operatorname{sign} f_j(\underline{\tau}_1, \lambda, \underline{\tau}_2) = \operatorname{sign} R(t)(t - x_0)^{k_0'} = (-1)^{l-1-j+k_0'} = (-1)^{l-j}$ if $j < l_0$ and $\operatorname{sign} f_j(\underline{\tau}_1, \lambda, \underline{\tau}_2) = \operatorname{sign} R(t) = (-1)^{l-1-j}$ if $j > l_0$. On the other side, if $(\underline{\tau}_1, \lambda, \underline{\tau}_2) \in B_j$, then this means that we move τ_j from a_j to a_{j+1} hence the sign of $R(t)$ changes. That is, the sign of $R(t)$ is the same as that of $f_j(\underline{\tau}_1, \lambda, \underline{\tau}_2)$, hence if $j < l_0$, then $\operatorname{sign} f_j(\underline{\tau}_1, \lambda, \underline{\tau}_2) = \operatorname{sign} R(t)(t - x_0)^{k_0'} = (-1)^{l-j+1}$ and if $j > l_0$, then $\operatorname{sign} f_j(\underline{\tau}_1, \lambda, \underline{\tau}_2) = (-1)^{l-j}$, which shows the sign change in both cases (when $j = 1, \ldots, l_0 - 1$ and when $j = l_0 + 1, \ldots, l$).

As regards $j = l_0$, we estimate $Z_1(t)$ and $R(t)$ first. Let $\rho_1 := 1/4 \min(a - a', x_0 - a, b - x_0, b' - b) > 0$. Considering $Z_1(t)$, it is easy to see that there exists $C_3 > 0$ such that for all $t \in [\alpha - \rho_1, \alpha + \rho_1] \cup [\beta - \rho_1, \beta + \rho_1]$ we have $1/C_3 \leq Z_1(t) \leq C_3$. The family of possible polynomials $R(\underline{\tau}; t)$ also has this property: there exists $C_4 > 0$ such that for any $(\underline{\tau}_1, \lambda, \underline{\tau}_2) \in \mathcal{R}$, and for any $t \in [\alpha - \rho_1, \alpha + \rho_1] \cup [\beta - \rho_1, \beta + \rho_1]$ we have $1/C_4 \leq |R(\underline{\tau}; t)(t - x_0)^{k_0'}| \leq C_4$. Now we need Nikolskii inequality to give a lower estimate for the integral of P_0 near α and β. Using that $\|P_0(\alpha, \mu; .)\|_{[\alpha-\rho_1, \alpha+\rho_1]} = P_0(\alpha) = 1$ and $\deg(P_0) = 2\mu$, Nikolskii inequality (see e.g. [14], p. 498, Theorem 3.1.4.) yields that there exists $C_5 > 0$ independent of μ and P_0 such that

$$\int_{\alpha-\rho_1}^{\alpha+\rho_1} P_0(\alpha, \mu; t) dt = \int_{\alpha-\rho_1}^{\alpha+\rho_1} |P_0(\alpha, \mu; t)|\, dt \geq C_5 \frac{1}{\mu^2}$$

with some $C_5 > 0$ depending on ρ_1 only and we can easily obtain

$$\int_{\alpha-\rho_1}^{\alpha+\rho_1} P_0(\alpha, \mu; t) Z_1(t) \left| R(\underline{\tau}; t)(t - x_0)^{k_0'} \right| dt \geq \frac{C_5}{C_3 C_4} \frac{1}{\mu^2} \tag{14}$$

as well. Moreover, for any $\lambda \in [0,1]$, $\max_{[\alpha-\rho_1, \alpha+\rho_1]} P_1(.) \geq 1 - \lambda$, hence applying Nikolskii inequality (see e.g. [14], p. 498, Theorem 3.1.4.) on these intervals,

$$\int_{\alpha-\rho_1}^{\alpha+\rho_1} P_1(\lambda, \mu; t) Z_1(t) \left| R(\underline{\tau}; t)(t - x_0)^{k_0'} \right| dt \geq \frac{C_5}{C_3 C_4} \frac{1 - \lambda}{\mu^2}$$

and similarly for $[\beta - \rho_1, \beta + \rho_1]$.

We need an upper estimate too. If $t \in [a_0, a_{l+1}]$, $|t-\alpha| \geq \rho_1$, then with $\rho_2 := 1 - \left(\frac{\rho_1}{c_2}\right)^2 < 1$ we can write

$$P_0(\alpha, \mu; t) \leq \rho_2^\mu$$

and if $t \in H := [a_0, \alpha - \rho_1] \cup [\alpha + \rho_1, \beta - \rho_1] \cup [\beta + \rho_1, a_{l+1}]$ then

$$P_0(\alpha, \mu; t) Z_1(t) \left| R(t)(t-x_0)^{k_0'} \right|, P_0(\beta, \mu; t) Z_1(t) \left| R(t)(t-x_0)^{k_0'} \right| \leq C_3 C_4 \rho_2^\mu \tag{15}$$

and

$$P_0(\alpha, \mu; t) Z_1(t) \left| R(t)(t-x_0)^{k_0'} \right| \leq C_3 C_4 \rho_2^\mu, \ |t-\beta| \leq \rho_1, \tag{16}$$

$$P_0(\beta, \mu; t) Z_1(t) \left| R(t)(t-x_0)^{k_0'} \right| \leq C_3 C_4 \rho_2^\mu, \ |t-\alpha| \leq \rho_1. \tag{17}$$

Now we can investigate $f_{l_0}(.)$ on $A_{l_0} := \{(\tau_1, \ldots, \tau_{l_0-1}, \lambda, \tau_{l_0+1}, \ldots, \tau_l) \in \mathcal{R} : \lambda = 0\}$: by (14) we can write

$$\left| \int_{a_{l_0}}^{x_0} P_0(\alpha, \mu; t) Z_1(t) R(\underline{\tau}; t)(t-x_0)^{k_0'} dt \right|$$
$$\geq \int_{\alpha-\rho_0}^{\alpha+\rho_0} P_0(\alpha, \mu; t) Z_1(t) \left| R(\underline{\tau}; t)(t-x_0)^{k_0'} \right| dt \geq \frac{C_5}{C_3 C_4} \frac{1}{\mu^2}$$

and by (15), we can write

$$\left| \int_{x_0}^{a_{l_0+1}} P_0(\alpha, \mu; t) Z_1(t) R(\underline{\tau}; t)(t-x_0)^{k_0'} dt \right| \leq c_2 C_3 C_4 \rho_2^\mu.$$

These last two displayed estimates show that $f_{l_0}(.)$ on A_{l_0} has the same sign as $R(t)(t-x_0)^{k_0'}$ on (a_{l_0}, x_0) (that is, $(-1)^{l-l_0-1+k_0'} = (-1)^{l-l_0}$) if μ is large ($\mu \geq \mu_1$). Similarly, by replacing α with β, we can say that $f_{l_0}(.)$ on $B_{l_0} := \{(\tau_1, \ldots, \tau_{l_0-1}, \lambda, \tau_{l_0+1}, \ldots, \tau_l) \in \mathcal{R} : \lambda = 1\}$ has the same sign as $R(t)(t-x_0)^{k_0'}$ on (x_0, a_{l_0+1}) (that is, $(-1)^{l-l_0+1}$), again if μ is large ($\mu \geq \mu_2$). These two observations show that on the opposite sides A_{l_0} and B_{l_0}, $f_{l_0}(.)$ has different signs (since k_0' is odd). Obviously, all $f_j(.)$ functions are continuous.

Now the conditions of Poincaré-Miranda theorem are satisfied, hence there exists $(\underline{\tau}_1, \lambda, \underline{\tau}_2) \in \mathcal{R}$ such that $f_j(\underline{\tau}_1, \lambda, \underline{\tau}_2) = 0$ for all $j = 1, \ldots, l$. Fix these values and denote them by the same letters in the rest of this proof.

Finally, in (13), we choose $C_1 \in \mathbf{R}$ so that $S(x_0) = 1$, where actually we can write

$$\frac{1}{C_1} = \int_{a_{l_0}}^{x_0} P_1(\lambda, \mu; t) Z_1(t)\, R(\underline{\tau}; t)(t-x_0)^{k_0'}\, dt$$

and by knowing the sign of $R(\underline{\tau}; .)$ over (a_{l_0}, x_0), $\operatorname{sign} C_1 = (-1)^{l-1-l_0+k_0'} = (-1)^{l-l_0}$ and by (14), $|C_1| = O(\mu^2)$.

So S is uniquely determined and it has the following properties. $S(a_j) = 0$ for all $j = 1, \dots, l$, hence by the key observation, (11) holds. By the normalization (5) is true. (6) is also true, because of (13). For simplicity, put

$$S_1(t) := C_1 Z_1(t) P_1(t) R(t) (t - x_0)^{k_0'}.$$

To see (7), (8), (10), and the first half of (9) (with 1 in place of $\min(1, |Z(x)|)$) first note that (15) implies that

$$\left| Z_1(t) P_1(\mu; t) R(t) (t - x_0)^{k_0'} \right| \le C_3 C_4 \rho_2^{\mu} \tag{18}$$

when $t \in H = [a_0, \alpha - \rho_1] \cup [\alpha + \rho_1, \beta - \rho_1] \cup [\beta + \rho_1, a_{l+1}]$. Moreover, let us remark that

$$|P_1(\mu; t)| \le \rho_2^{\mu} \tag{19}$$

for $t \in H$. Let us choose $\delta_1 > 0$ such that $0 < \delta_1 < -1/64 \log(\rho_2)$, hence for large μ, $\mu \ge \mu_3$, we have

$$C_3 C_4 \rho_2^{\mu} \le \exp(-\delta_1(64\mu)).$$

Now, if $\mu \ge \mu_4$ is large enough and using $|C_1| = O(\mu^2)$, we can write

$$|S_1(t)| \le C_1 C_3 C_4 \rho_2^{\mu} \le \exp(-\delta_1(32\mu)), \quad t \in H.$$

Integrating this on $[a_1, x]$, $x \le \alpha - \rho_1$, we obtain for large μ, $\mu \ge \mu_5$, that

$$|S(x)| = \left| \int_{a_1}^{x} S_1(t) dt \right| \le c_2 C_1 C_3 C_4 \rho_2^{\mu} \le \exp(-\delta_1(16\mu))$$

moreover this also holds when $x \in [a_0, a_1]$. If $x \in [\alpha + \rho_1, x_0]$, then using that $S_1(t) \ge 0$ when $t \in [\alpha + \rho_1, x_0]$, we can write

$$\begin{aligned} 1 - \exp(-\delta_1(16\mu)) \le 1 - c_2 C_1 C_3 C_4 \rho_2^{\mu} \le \int_{a_1}^{x_0} S_1(t) dt - \int_{x}^{x_0} S_1(t) dt \\ = S(x) \le S(x_0) = 1. \end{aligned}$$

Similarly when $x \in [x_0, \beta - \rho_1]$, $S_1(t) \le 0$ on $[x_0, \beta - \rho_1]$, hence

$$\begin{aligned} 1 - \exp(-\delta_1(16\mu)) \le 1 - c_2 C_1 C_3 C_4 \rho_2^{\mu} \le \int_{a_1}^{x_0} S_1(t) dt + \int_{x_0}^{x} S_1(t) dt \\ = S(x) \le S(x_0) = 1. \end{aligned}$$

As for $[\beta + \rho_1, a_{l+1}]$, we know that $|S_1(t)| \le C_1 C_3 C_4 \rho_2^{\mu} \le \exp(-\delta_1(32\mu))$, and $S(a_{l_0+1}) = 0$, so for $x \in [a_{l_0+1}, a_{l+1}]$, $S(x) = \int_{a_1}^{x} S_1(t) dt = \int_{a_{l_0+1}}^{x} S_1(t) dt$ and $|S(x)| \le c_2 C_1 C_3 C_4 \rho_2^{\mu} \le \exp(-\delta_1(16\mu))$. For $x \in [\beta + \rho_1, a_{l_0+1}]$, we know that

$$\begin{aligned} S(x) &= \int_{a_1}^{x} S_1(t) dt = \int_{a_1}^{a_{l_0+1}} S_1(t) dt - \int_{x}^{a_{l_0+1}} S_1(t) dt \\ &= 0 + \int_{x}^{a_{l_0+1}} -S_1(t) dt = \int_{x}^{a_{l_0+1}} |S_1(t)| \, dt \le c_2 C_1 C_3 C_4 \rho_2^{\mu} \le \exp(-\delta_1(16\mu)). \end{aligned}$$

These last four displayed estimates show that (8) and first half of (9) hold since

$$\exp(-\delta_1(16\mu)) \le \exp(-2\delta_1(3\deg S))$$

if $\mu \ge \mu_6$ is large. (10) and (7) are also true, since $S'(.) = S_1(.)$ is nonnegative on (a_{l_0}, x_0) and is nonpositive on (x_0, a_{l_0+1}).

To establish the second half of (9) (with $Z(x)$ in place of $\min(1, |Z(x)|)$), we write (similarly to (18))

$$\begin{aligned} |S(x)| &= \left| C_1 \int_{a_1}^{x} Z_1(t) \frac{P_1(t)}{\|P_1\|_H} R(t)(t-x_0)^{k_0'} dt \right| \|P_1\|_H \\ &\le |C_1| \int_{a_1}^{x} Z_1(t) \frac{|P_1(t)|}{\|P_1\|_H} |R(t)||t-x_0|^{k_0'} dt \|P_1\|_H \\ &\le |C_1| C_4 \int_{a_j}^{x} Z_1(t) dt \, \|P_1\|_H \end{aligned}$$

where $x \in [a_j, a_{j+1}]$ and $H = [a_0, \alpha - \rho_1] \cup [\alpha + \rho_1, \beta - \rho_1] \cup [\beta + \rho_1, a_{l+1}]$. It is easy to see that

$$\frac{\int_{a_j}^{x} Z_1(t) dt}{|Z(x)|}$$

has finite limit as $x \to a_j$ since Z and Z_1 have zeros of order k_j and k_j' at a respectively. The same is true on the left hand side neighborhood of a_j. Hence we see that $\int_{a_1}^{x} Z_1(t)dt/|Z(x)|$ is bounded when $x \in H$, so, using $\|P_1\|_H \le \exp(-\delta_1 32\mu)$ coming from (19), we obtain that the second half of (9) holds for large μ, $\mu \ge \mu_7$.

To fulfill (12), consider $Q := S^2$. Then, the degree of Q is $2(k_1' + \ldots + k_l' + l - 2 + k_0' + 2\mu + 1) = 4\mu + const$. By squaring S defined in (13), it is easy to see that (5), (6), (7), (9), (11) and (10) are preserved, and actually, (8) too:

$$(1 - \exp(-2\delta_1(3\deg S)))^2 \ge 1 - \exp(-2\delta_1 \deg Q)$$

since $2\exp(-2\delta_1 3\deg S) - \exp(-4\delta_1 3\deg S) \le \exp(-2\delta_1 \deg Q)$ if $\deg S$ is large (that is, if $\mu \ge \mu_8$).

Finally, we have a sequence of polynomials for particular degrees. The basic idea to use the same polynomial for larger degree works now, because of the following. Put $m_1(m) := \max\{m_1 : m_1 = 4\mu + 2(k_1' + \ldots + k_l' + l - 2 + k_0' + 1), m_1 \le m, \mu \in \mathbf{N}\}$. For general $m \in \mathbf{N}$, replacing the error term for m from $m_1(m)$ brings in a factor $\exp(-2\delta_1 m)/\exp(-2\delta m_1(m))$ which can be estimated as

$$\limsup_{m \to \infty} \exp(-2\delta_1 m)/\exp(-2\delta_1 m_1(m)) = \exp(-2\delta_1 const) < 1,$$

where *const* is actually $2(k_1' + \ldots + k_l' + l - 2 + k_0' + 1)$. Hence, if $\mu \ge \mu_9$ is large, then

$$\exp(-2\delta_1 m_1(\deg Q)) \le \exp(-\delta_1 \deg Q)$$

which finishes the proof. □

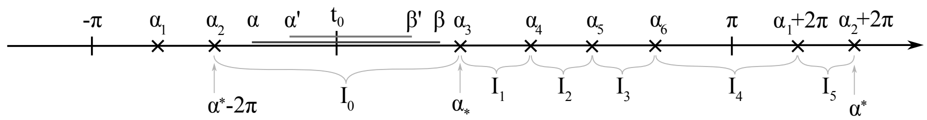

Figure 1: Prescribed zeros and intervals in the trigonometric case

Remark: Note that (the second half of) (9) implies (11).

We need the following trigonometric form of fast decreasing polynomials. In the proof we use so-called half-integer trigonometric polynomials $\sum_{j=0}^{n} a_j \cos((j+1/2)t)+b_j \sin((j+1/2)t)$. They are natural in this context, see, e.g. the product representation [2], p. 10, or Videnskii's original paper [30], or the paper [16].

Theorem 3. *Let $t_0, \alpha, \beta, \alpha', \beta' \in (-\pi, \pi)$ be such that $-\pi < \alpha' < \alpha < t_0 < \beta < \beta' < \pi$ and $\alpha_1, \dots, \alpha_l \in (-\pi, \pi) \setminus [\alpha', \beta']$ be with the corresponding positive integer powers $k_1, \dots, k_l$. Put $Z(t) := \prod_{j=1}^{l} \left|\sin \frac{t-\alpha_j}{2}\right|^{k_j}$.*

Then there exists $\delta_1 > 0$ such that for all large m there exists a trigonometric polynomial Q_m with degree at most m such that

$$Q_m(t_0) = 1, \tag{20}$$

$$0 \le Q_m(t) < 1 \text{ for } t \in [-\pi, \pi), t \ne t_0, \tag{21}$$

$$Q_m^{(k)}(\alpha_j) = 0, \quad j = 1, \dots, l, \;\; k = 0, 1, \dots, k_j, \tag{22}$$

$$Q_m(t) \le \min(1, |Z(t)|) \exp(-\delta_1 m) \text{ for } t \in [-\pi, \pi] \setminus (\alpha', \beta'), \tag{23}$$

$$|Q_m(t) - 1| \le \exp(-\delta_1 m) \text{ for } t \in [\alpha, \beta], \tag{24}$$

$$Q_m(t) \text{ is strictly monotone on } [\alpha', \alpha] \text{ and on } [\beta, \beta']. \tag{25}$$

Proof. Briefly, we use similar idea as in the previous proof (Theorem 2), but there are lots of differences.

First, we introduce the intervals between the neighboring α_j's as follows using the ordering of $\alpha_j + \epsilon_j 2\pi$, $j = 1, \dots, l$, and $\epsilon_j = 0$ if $\alpha_j > \beta'$ and $\epsilon_j = 1$ otherwise. Let I_j's, $j = 1, \dots, l-1$ denote the closed intervals such that endpoints are the $\alpha_j + \epsilon_j 2\pi$'s and they are disjoint except for the endpoints, and they are ordered from left to right (that is, if $t_1 \in I_j$ and $t_2 \in I_k$ and $j \le k$, then $t_1 \le t_2$). Denote the left endpoint of I_1 by α_*, and the right endpoint of I_{l-1} by α^*, that is, α_* and α^* are the minimum and maximum of $\alpha_j + \epsilon_j 2\pi$'s respectively. Put $I_0 := [\alpha^* - 2\pi, \alpha_*]$, this way $I_0, I_1, \dots, I_{l-1}$ cover an interval of length 2π and $t_0 \in I_0$, $[\alpha', \beta'] \subset I_0$. Note that I_j's are not necessarily subsets of $(-\pi, \pi)$.

We define

$$\tilde{Z}_1(t) := \prod_{j=1}^{l} \left(\sin\frac{t-\alpha_j}{2}\right)^{k_j'}, \quad \tilde{R}(\underline{\tau};t) = \tilde{R}(t) := \prod_{j=1}^{l-1} \sin\frac{t-\tau_j}{2},$$

$$\tilde{P}_0(t) = \tilde{P}_0(a,\mu;t) := \left(\cos\frac{t-a}{2}\right)^{2\mu},$$

$$\tilde{P}_1(a,b,\lambda,\mu;t) = (1-\lambda)\tilde{P}_0(a,\mu;t) + \lambda\tilde{P}_0(b,\mu;t)$$

where $k_j' = k_j$ if k_j is even and $k_j' = k_j + 1$ if k_j is odd, for $j = 1,\ldots,l$, and $\tau_j \in I_j$, $j = 1,\ldots,l-1$, and $\alpha' < a < \alpha < \beta < b < \beta'$, $a := (\alpha+\alpha')/2$, $b := (\beta+\beta')/2$, and $\lambda \in [0,1]$. We also put $k_0' = k_0$ if k_0 is odd and $k_0' = k_0 + 1$ if k_0 is even; and $\underline{\tau} := (\tau_1,\ldots,\tau_{l-1})$. As above, if some of the parameters are fixed or unimportant in the current consideration, then we leave them out, e.g. $\tilde{P}_0(t) = \tilde{P}_0(a,\mu;t)$ and $\tilde{P}_1(t) = \tilde{P}_1(\mu;t) = \tilde{P}_1(\lambda,\mu;t) = \tilde{P}_1(a,b,\lambda,\mu;t)$.

Some immediate properties are the following: $\tilde{Z}(t)$, $\tilde{P}_0(t)$ and $\tilde{P}_1(t)$ are nonnegative trigonometric polynomials. If l is even, then $\tilde{R}(t)$ is a half-integer trigonometric polynomial, if l is odd, then it is a trigonometric polynomial (with degree $(l-1)/2$).

Consider

$$\tilde{S}_1(t) := \tilde{Z}_1(t)\,\tilde{P}_1(\mu,\lambda;t)\,\tilde{R}(\underline{\tau};t)\left(\sin\frac{t-t_0}{2}\right)^{k_0'}$$

which is a trigonometric polynomial if l is even and is a half-integer trigonometric polynomial if l is odd. We need

$$\tilde{S}_2(t) := \begin{cases} \tilde{S}_1(t), & \text{if } l \text{ is even,} \\ \tilde{S}_1(t)\cos\frac{t-(\alpha^*-\pi)}{2}, & \text{if } l \text{ is odd} \end{cases}$$

which is a trigonometric polynomial in both cases.

Now we would like to integrate $\tilde{S}_1(.)$ and get a trigonometric polynomial too. To do this, we use Poincaré-Miranda theorem, as in the proof of Theorem 2. Consider the rectangle $\mathcal{R} := [0,1] \times I_1 \times I_2 \times \ldots \times I_{l-1}$ and $(\lambda,\underline{\tau}) = (\lambda,\tau_1,\ldots,\tau_{l-1}) \in \mathcal{R}$. We use the functions

$$f_j(\lambda,\underline{\tau}) := \int_{I_j} \tilde{S}_2(\lambda,\underline{\tau},\mu;t)dt, \; j = 0,1,\ldots,l-1.$$

Note that $\sin\frac{t-t_0}{2}$ is negative on $(\alpha^*-2\pi,t_0)$ and is positive on (t_0,α^*), $\cos\frac{t-(\alpha^*-\pi)}{2}$ is positive on $(\alpha^*-2\pi,\alpha^*)$ but it introduces an extra zero at α^*. It can be verified same way as in the proof of Theorem 2 that there are sign changes in f_0 as λ changes from 0 to 1, and in f_j as τ_j goes from the left endpoint of I_j to the right endpoint of I_j.

Poincaré-Miranda theorem shows that there are particular $\lambda \in [0,1]$, $\tau_1 \in I_1,\ldots,\tau_{l-1} \in I_{l-1}$ such that all the f_j's are zero; fix this solution and denote it by $\lambda, \tau_1, \ldots, \tau_{l-1}$ in the rest of this proof. Summing up these integrals for all $j = 0, 1, \ldots, l-1$, we also obtain that $\int_{\alpha^*-2\pi}^{\alpha^*} \tilde{S}_2(t)dt = 0$.

Put

$$\tilde{S}(t) := \int_{\alpha_*}^{t} C_1 \tilde{S}_2(\tau) d\tau$$

where C_1 will be chosen later (like in the proof of Theorem 2). In both cases (l is even or odd), the integrand is a real trigonometric polynomial. Since the integral of $\tilde{S}_2(t)$ over $[\alpha^* - 2\pi, \alpha^*]$ is 0, $\tilde{S}(t)$ is also a trigonometric polynomial. C_1 can be chosen so that

$$\int_{\alpha_*}^{t_0} C_1 \tilde{S}_2(t) dt = 1$$

holds. The properties (20), (22), (23), (24) and (25) can be verified same way as in the proof of Theorem 2. A key tool was the Nikolskii inequality for algebraic polynomials and it should be replaced with the similar inequality for trigonometric polynomials, which is again due to Nikolskii (see, e.g [14], p. 495, Theorem 3.1.1). Again, squaring $\tilde{S}$, we can construct the trigonometric polynomial which also satisfies (21).

□

3.2 Rough Markov- and Bernstein-type inequalities

The following two propositions have rather simple proofs, they may be known, but we could not find reference for them.

Proposition 4. *Let $I \subset (-\pi, \pi)$ be a closed set consisting of finitely many disjoint intervals such that none of them is a singleton and k be a positive integer. Then there exists $C = C(I, k) > 0$ such that for all trigonometric polynomial T_n with degree n, we have*

$$\left\| T_n^{(k)} \right\|_I \leq C n^{2k} \left\| T_n \right\|_I . \tag{26}$$

This immediately follows from iterating Videnskii's inequality on each component (maximal subinterval) of I. For Videnskii's inequality, see [2], p. 243 (Exercise E.19 part c]) or [31].

We also need a rough Bernstein-type inequality for higher derivatives of trigonometric polynomials.

Proposition 5. *Let $I \subset (-\pi, \pi)$ be again a closed set consisting of finitely many disjoint intervals such that none of them is a singleton and k be a positive integer. Fix a closed set $I_0 \subset \operatorname{Int} I$ (subset of the one dimensional interior of I). Then there exists $C = C(I, I_0, k) > 0$ such that for all trigonometric polynomial T_n with degree n, we have for $t \in I_0$*

$$\left| T_n^{(k)}(t) \right| \leq C n^{k} \left\| T_n \right\|_I . \tag{27}$$

This again, immediately follows from applying Videnskii's inequality (see [2], p. 243, E.19 part b]) iteratively on the component (say I_0^+) of I containing I_0 and finally using $\|T_n\|_{I_0^+} \leq \|T_n\|_I$.

3.3 Asymptotically sharp Markov-type inequality

Theorem 6. *Let $E \subset (-\pi, \pi)$ be a compact set satisfying (2). Then for any trigonometric polynomial T_n with degree n, we have*

$$\left\| T_n^{(k)} \right\|_{[a-\rho,a]} \le (1+o(1)) n^{2k} \Omega(E_{\mathbb{T}}, e^{ia})^{2k} \frac{8^k \pi^{2k}}{(2k-1)!!} \|T_n\|_E \tag{28}$$

where $o(1)$ is an error term that tends to 0 as $n \to \infty$, depends on E and a, but it is independent of T_n. This inequality is sharp, that is, there is a sequence of trigonometric polynomials T_n, $n = 1, 2, \ldots$, such that $\deg T_n = n$ and

$$\left| T_n^{(k)}(a) \right| \ge (1-o(1)) n^{2k} \Omega(E_{\mathbb{T}}, e^{ia})^{2k} \frac{8^k \pi^{2k}}{(2k-1)!!} \|T_n\|_E \,, \tag{29}$$

where $o(1) \to 0$ is an error term depending on E and n.

Proof. The proof of (28) is divided into five steps and then (29) will be established.

First step. We prove the assertion when E is a T-set, and T_n is polynomial of the defining polynomial U_N for this set. That is, $E = \{t \in (-\pi, \pi) : |U_N(t)| \le 1\}$ (as in (4)) and there is a real, algebraic polynomial P such that $T_n(t) = P(U_N(t))$. We may assume that $U_N(a) = 1$ (we know that $|U_N(a)| = 1$).

Now we use Faà di Bruno's formula (1). Note that, in our setting $f = P$ (outer function) and $g = U_N$ (inner function), hence the product is independent of P and n (and T_n too). Hence we reorder the terms decreasingly:

$$(P \circ U_N)^{(k)}(a) = P^{(k)}(1) \left(U_N'(a)\right)^k + \ldots \tag{30}$$

where in the remaining terms only $P^{(k-1)}(1)$, $P^{(k-2)}(1)$, ... $P'(1)$ occur. There are finitely many remaining terms and by (26), they grow like n^{2k-2} as $n \to \infty$. As for the first term, we can use the classical V. Markov inequality (see e.g. [2], p. 254) and $\|P\|_{[-1,1]} = \|T_n\|_E$, hence with $d := \deg(P)$,

$$|P'(1)| \le \frac{d^2(d^2-1)\ldots\left(d^2-(k-1)^2\right)}{(2k-1)!!} \|T_n\|_E \le d^{2k} \frac{1}{(2k-1)!!} \|T_n\|_E$$

where actually

$$\frac{d^2(d^2-1)\ldots\left(d^2-(k-1)^2\right)}{d^{2k}} \to 1 \tag{31}$$

as $n \to \infty$ (which is equivalent to $d \to \infty$).

As for $U_N'(a)$, we use the density of the equilibrium measure, more precisely formula (3.21) from [28] (and $a = a_{2j_0}$), hence

$$|U_N'(a)| = 2N^2 \frac{\prod_{l=1}^m \left|e^{ia} - e^{i\tau_l}\right|^2}{\prod_{l=1,\ldots,2m, l\ne 2j_0} |e^{ia} - e^{ia_l}|} = 2N^2 M_{a,k} = 8\pi^2 N^2 \Omega(E_{\mathbb{T}}, e^{ia})^2. \tag{32}$$

Putting these together:

$$\left|T_n^{(k)}(a)\right| \le (1+o(1))8^k\pi^{2k}\frac{1}{(2k-1)!!}\Omega(E_{\mathbb{T}},e^{ia})^{2k}\ n^{2k}\left\|T_n\right\|_E .$$

Now we extend the previous inequality from a to $[a-\rho,a]$ (as in (28)). Basically we use the smaller growth of the rough Bernstein-type inequality (27) and the continuity of U_N'. For any $\varepsilon>0$, we can select $\eta>0$ such that $[a-\eta,a]\subset E$ and for $t\in[a-\eta,a]$ it is true that

$$|U_N'(t)| \le (1+\varepsilon)|U_N'(a)| = (1+\varepsilon)8\pi^2N^2\Omega(E_{\mathbb{T}},e^{ia})^2.$$

Then for $t\in[a-\eta,a]$ we get from (30) and again from (26) that

$$\left|T_n^{(k)}(t)\right| \le (1+o(1))(1+\varepsilon)8^k\pi^{2k}\frac{1}{(2k-1)!!}\Omega(E_{\mathbb{T}},e^{ia})^{2k}\ n^{2k}\left\|T_n\right\|_E . \tag{33}$$

Now, on $[a-\rho,a-\eta]$ (if not empty), we can use the rough Bernstein-type inequality (27), hence we obtain an upper estimate for $T^{(k)}(t)$ which has growth order n^k, which is smaller than n^{2k}, the growth order of the Markov factor. So if n is large (depending on ε), then (33) holds for $t\in[a-\rho,a-\eta]$ too. Now letting $\varepsilon\to 0$ appropriately, (28) follows for $T_n(.)=P(U_N(.))$ as $d=\deg(P)\to\infty$.

Second step. Now we establish (28) when E is a T-set and T_n is arbitrary trigonometric polynomial. We use symmetrization here (see, [25] pp. 151-152 and [28], pp. 2997-2998, including Lemma 3.2) and fast decreasing trigonometric polynomials (see Subsection 3.1). In this step we work in a smaller neighborhood of a, i.e. on $[a-\rho_0,a]$ where $\rho_0<\rho$ is defined later.

Let j_0 correspond to the interval in which a is. More precisely, since E is a T-set in this case, there are $2N$ disjoint, open intervals such that U_N maps these intervals to $(-1,1)$ in a bijective way. Let us label them by $E_j=(\alpha_{2j-1},\alpha_{2j})$ where $-\pi<\alpha_1<\alpha_2\le\alpha_3<\alpha_4\le\ldots\le\alpha_{2N-1}<\alpha_{2N}<\pi$. Hence $a\in[\alpha_{2j_0-1},\alpha_{2j_0}]$ and by (2), $a=\alpha_{2j_0}$. Put $\rho_0:=1/4\min(\alpha_{2j_0}-\alpha_{2j_0-1},\alpha_{2j_0+1}-\alpha_{2j_0},\rho,\pi/4)$.

We also need the following facts on T-sets. Since $U_N(.)$ is $2N$-to-1 mapping, we need its restricted inverses. Let $U_{N,j}^{-1}(t)$ be the inverse of U_N restricted to $[\alpha_{2j-1},\alpha_{2j}]$ and put $t_j(t)=t_j:=U_{N,j}^{-1}(U_N(t))$. Obviously, t_j is C^∞ on $\cup_{j=1}^N(\alpha_{2j-1},\alpha_{2j})$ and now we give estimates for the l-th derivative of $t_j(t)$, especially, as t approaches a. Similarly, as in [23], if $l=1$ or $l=2$, then

$$\frac{dt_j}{dt}=\frac{d}{dt}U_{N,j}^{-1}(U_N(t))=\frac{U_N'(t)}{U_N'(U_{N,j}^{-1}(U_N(t)))}=\frac{U_N'(t)}{U_N'(t_j)},$$

$$\begin{aligned}\frac{d^2}{dt^2}U_{N,j}^{-1}(U_N(t)) &= \frac{-\left(U_N'(t)\right)^2U_N''(U_{N,j}^{-1}(U_N(t)))}{\left(U_N'(U_{N,j}^{-1}(U_N(t)))\right)^3}+\frac{U_N''(t)}{U_N'(U_{N,j}^{-1}(U_N(t)))}\\ &= \frac{-(U_N'(t))^2U_N''(t_j)}{\left(U_N'(t_j)\right)^3}+\frac{U_N''(t)}{U_N'(t_j)}\end{aligned}$$

and for general l, Faà di Bruno's formula (1) implies that there is a universal polynomial Q_l (independent of U_N, depending on l only) which is a polynomial in $U_N^{(k)}(t)$ and $U_N^{(k)}(t_j)$ $k = 1, \dots, l$, that is $Q_l = Q_l(\dots, U_N^{(k)}(t), \dots, U_N^{(k)}(t_j), \dots)$ such that

$$\frac{d^l}{dt^l} U_{N,j}^{-1}(U_N(t)) = \frac{Q_l}{(U_N'(t_j))^{2l-1}}. \tag{34}$$

Here, Q_l is independent of n and T_n, hence $|Q_l| \le C$ for some $C = C(k, U_N) > 0$.

Moreover, we need to estimate $|U_N'(t_j)|$ as $t \to a$ and we split the argument into two cases. If j is such that $a_j \in \operatorname{Int} E$, that is, $U_N'(a_j) = 0$, and using that all the zeros of U_N are simple, we can infer that $U_N''(a_j) \ne 0$, so $|U_N'(t_j)| \ge O(|t_j - a_j|)$. On the other hand, if j is such that $a_j \in E \setminus \operatorname{Int} E$, that is, $U_N'(a_j) \ne 0$, then simply $U_N'(t_j) \approx U_N'(a_j)$. Hence, in any case

$$|U_N'(t_j)| \ge O(|t_j - a_j|). \tag{35}$$

For an arbitrary polynomial T_n consider $V_n(t) = L_{\sqrt{n}}(t) T_n(t)$, where $L_{\sqrt{n}}(.)$ denotes the fast decreasing polynomial which has the following properties. $L_{\sqrt{n}}(.)$ has degree at most $\sqrt{n}$, it is a fast decreasing trigonometric polynomial and peaking at a very smoothly (that is, $L_{\sqrt{n}}(a) = 1$ and $L_{\sqrt{n}}^{(j)}(a) = 0$, $j = 1, 2, \dots, 2k^2$), $L_{\sqrt{n}}(.)$ is approximately 1 on $[a - \rho_0, a + \rho_0]$ and is approximately 0 outside $[a - 2\rho_0, a + 2\rho_0]$ and vanishes at the other extremal points of U_N up to order $2k^2$ (that is, if $U_N(t) = \pm 1$, $t \ne a$, then $L_{\sqrt{n}}^{(j)}(t) = 0$, $j = 0, 1, \dots, 2k^2$). Such polynomial $L(.) = L_{\sqrt{n}}(.)$ exists because of Theorem 3.

For simplicity, put $W(t) := \prod_j \left(\sin \frac{t - \alpha_j}{2} \right)^{2k}$ where $j = 1, \dots, 2N$, $j \ne j_0$. This W is a nonnegative trigonometric polynomial and has sup norm at most 1. There is another trigonometric polynomial $Y(.)$ such that

$$L(t) = Y(t) W^k(t).$$

The sup norm of Y over $[-\pi, a - \rho_0] \cup [a + \rho_0, \pi]$ can be estimated using (23) with W^k in place of Z. Hence, for $t \in [-\pi, a - \rho_0] \cup [a + \rho_0, \pi]$

$$|Y(t)| = \left| \frac{L(t)}{W^k(t)} \right| \le \min \left(\frac{1}{W^k(t)}, 1 \right) \exp\left(-(\deg L) \delta_1 \right).$$

Differentiating $L(.)$ j-times, $j = 0, 1, \dots, k$ we write

$$L^{(j)}(t) = \sum_{l=0}^{j} \binom{j}{l} Y^{(j-l)}(t) \left(W^k \right)^{(l)}(t). \tag{36}$$

Here $\left(W^k \right)^{(l)}(t) = W(t) \cdot \dots$ where $W(t)$ is multiplied with other terms depending on $W, W', \dots, W^{(l)}$, k and α_j's only, and it is independent from n and T_n. As regards $Y^{(j-l)}(t)$, we can use Videnskii's inequality for $Y(.)$ on

$[-\pi, a-\rho_0] \cup [a+\rho_0, \pi]$ (which is actually an interval on the torus), so there exists a $C>0$ such that for all $t \in [-\pi, a-2\rho_0] \cup [a+2\rho_0, \pi]$ and all $l = 0, 1, \ldots, j$

$$\left| Y^{(j-l)}(t) \right| \leq C \left(\deg Y\right)^{j-l} \exp\left(-(\deg L)\delta_1\right). \tag{37}$$

Summing up these estimates as in (36), we can write with $\deg L \leq \sqrt{n}$

$$\left| L^{(j)}(t) \right| \leq C W(t) n^{j/2} \exp\left(-\sqrt{n}\delta_1\right) \tag{38}$$

where $C>0$ is independent of n and T_n and $t \in [-\pi, a-2\rho_0] \cup [a+2\rho_0, \pi]$.

This V_n has degree at most $n+\sqrt{n}$ and satisfies

$$\left.\begin{aligned} \|V_n\|_E &\leq \|T_n\|_E\,, \\ V_n(t) &= \left(1+O(\beta^{\sqrt{n}})\right) T_n(t) \text{ for } t \in [a-\rho_0, a], \\ |V_n(t)| &= O(\beta^{\sqrt{n}}) \|T_n\|_E \text{ for } t \in E \setminus [a-2\rho_0, a] \end{aligned}\right\} \tag{39}$$

where $\beta = \exp(-\delta_1) < 1$.

Now, (by Leibniz formula), for all $l = 1, \ldots, k$

$$V_n^{(l)}(t) - T_n^{(l)}(t) = \left(L_{\sqrt{n}}(t) - 1\right) T_n^{(l)}(t) + \sum_{j=1}^{l} \binom{l}{j} L_{\sqrt{n}}^{(j)}(t) T_n^{(l-j)}(t). \tag{40}$$

Using the rough Markov-type inequality (26), there exists a constant $C = C(E,k) > 0$ such that for all $1 \leq j \leq k$, $t \in E$

$$\begin{gathered} \left| L_{\sqrt{n}}^{(j)}(t) \right| \leq C \sqrt{n}^{2j} \left\| L_{\sqrt{n}} \right\|_E = C n^j, \\ \left| T_n^{(j)}(t) \right| \leq C n^{2j} \|T_n\|_E \end{gathered}$$

and if $t \in E \setminus [a-2\rho, a]$, then applying (26) for $L_{\sqrt{n}}$ on $E \setminus [a-2\rho, a]$, we can write

$$\left| L_{\sqrt{n}}^{(j)}(t) \right| \leq C \sqrt{n}^{2j} \left\| L_{\sqrt{n}} \right\|_{E \setminus [a-2\rho, a]} = C n^j \beta^{\sqrt{n}}. \tag{41}$$

These imply that for $l = 1, \ldots, k$

$$\left| V_n^{(l)}(t) - T_n^{(l)}(t) \right| = O\left(n^{2l}\beta^{\sqrt{n}} + n^{2l-1}\right) \|T_n\|_E\,,\ t \in [a-\rho, a] \tag{42}$$

and

$$\left| V_n^{(l)}(t) \right| = O\left(n^{2l}\beta^{\sqrt{n}}\right) \|T_n\|_E\,,\ t \in E \setminus [a-2\rho, a].$$

Define the "symmetrized" polynomial as

$$T^*(t) := \sum_{j=1}^{N} V_n\left(t_j\right).$$

This T^* will be algebraic polynomial of $U_N(.)$, see Lemma 3.2 in [28], and $\deg(T^*) \le n + \sqrt{n} = (1 + o(1))n$.

Now we compare $(T^*)^{(k)}(t)$ with $T_n^{(k)}(t)$ when $t \in [a - \rho, a]$. If $j = j_0$, that is, $t_j = t$, then $V_n(t_j) = V_n(t)$, and we can apply (42) (when $l = k$). If $j \ne j_0$, then we would like to show that $\left|\frac{d^k}{dt^k} V_n(t_j)\right|$ is small. We use (40) first so

$$\left|\frac{d^k}{dt^k} V_n(t_j)\right| \le \sum_{l=0}^{k} \binom{k}{l} \left|\frac{d^l}{dt^l} L_{\sqrt{n}}\left(U_{N,j}^{-1}(U_N(t))\right)\right| \left|\frac{d^{k-l}}{dt^{k-l}} T_n\left(U_{N,j}^{-1}(U_N(t))\right)\right| \tag{43}$$

which we continue later. For the second factor, we use (1) again with similar groupings of the terms as in (30), because the first term involves $\frac{d^{k-l}}{dt^{k-l}} T_n$ (at t_j) and all the other terms involve lower derivatives of T_n. So we can write, with the help of (26), and (34), (35)

$$\begin{aligned}\left|\frac{d^{k-l}}{dt^{k-l}} T_n\left(U_{N,j}^{-1}(U_N(t))\right)\right| &\le \left|T_n^{(k-l)}(t_j)\right| \left|\frac{d}{dt} U_{N,j}^{-1}(U_N(t))\right|^{k-l} + |\ldots| \\ &\le C n^{2k-2l} \frac{1}{|t_j - a_j|^{2(k-l)-1}} \|T_n\|_E .\end{aligned}$$

Now we use the zeros of $L_{\sqrt{n}}(.)$ (and $W(t)$) to get rid of the factors $1/|t_j - a_j|^{2(k-l)-1}$. To estimate the first factor on rhs of (43), we use (1) for $L_{\sqrt{n}}(.)$ and $U_{N,j}^{-1}(U_N(t))$ with (41) (since $t_j \notin [a - 2\rho, a]$) and (38). Hence

$$\begin{aligned}\left|\frac{d^l}{dt^l} L_{\sqrt{n}}\left(U_{N,j}^{-1}(U_N(t))\right)\right| &\le C\sqrt{n}^{2l} \beta^{\sqrt{n}} |W(t_j)|\, n^{2l} \frac{1}{|t_j - a_j|^{2l-1}} \|T_n\|_E \\ &= C n^{3l} \beta^{\sqrt{n}} \frac{|W(t_j)|}{|t_j - a_j|^{2l-1}} \|T_n\|_E\end{aligned}$$

and using that a_j is a zero of W (of order k), the fraction $|W(t_j)|/|t_j - a_j|^{2l-1}$ is actually bounded.

Multiplying together the last two displayed estimates and using that $|W(t_j)|/|t_j - a_j|^{2k}$ is bounded (independently of t, j and n), we can continue (43),

$$\le \sum_{l=0}^{k} \binom{k}{l} C n^l \beta^{\sqrt{n}} n^{2k-2l} \|T_n\|_E \le C n^{2k} \beta^{\sqrt{n}} \|T_n\|_E .$$

Collecting all the calculations in this paragraph, for $t \in [a - \rho, a]$ we can write

$$\left|(T^*)^{(k)}(t) - T_n^{(k)}(t)\right| \le O\left(n^{2k} \beta^{\sqrt{n}}\right) \|T_n\|_E . \tag{44}$$

Comparing the sup norms of T_n and T^*, we split the estimate into two cases (see also (39)). If $t \in E \setminus [a - 2\rho_0, a]$, then

$$|T^*(t)| \le \sum_{j=1}^{N} \left|L_{\sqrt{n}}(t_j)\right| |T_n(t_j)| \le N C \beta^{\sqrt{n}} \|T_n\|_E = o(1) \|T_n\|_E .$$

If $t \in [a - 2\rho_0, a]$, then

$$
\begin{aligned}
|T^*(t)| &\le \left|L_{\sqrt{n}}(t)\right| |T_n(t)| + \sum_{j \ne j_0} \left|L_{\sqrt{n}}(t_j)\right| |T_n(t_j)| \\
&\qquad\qquad \le \left(1 + NC\beta^{\sqrt{n}}\right) \|T_n\|_E = (1 + o(1)) \|T_n\|_E .
\end{aligned}
$$

These two estimates yield

$$
\|T^*\|_E \le (1 + o(1)) \|T_n\|_E . \tag{45}
$$

Applying (44), (45) and the previous case for T^* (when T^* is a polynomial of U_N), we obtain (28) for T-sets and for arbitrary polynomials.

Third step. Now let E be an arbitrary set consisting of finite number of intervals: $E = \cup_{j=1}^m [a_{2j-1}, a_{2j}]$. Using the density of T-sets (see Section 2.1), there is a T-set E' such that $E' \subset E$, $a \in E'$ and

$$
\Omega(E_{\mathbb{T}}, e^{ia}) \le \Omega(E'_{\mathbb{T}}, e^{ia}) \le (1 + \varepsilon)\Omega(E_{\mathbb{T}}, e^{ia})
$$

where $\varepsilon > 0$ is arbitrary and $E' = E'(E, \varepsilon)$. Here the first inequality comes from the monotonicity of $\Omega(.,.)$ (and from $E' \subset E$) and the second comes from the density result. Obviously, $\|T_n\|_{E'} \le \|T_n\|_E$. Now, applying the previous step (for arbitrary polynomials on T-sets), we can write for $t \in [a - \rho, a]$

$$
\begin{aligned}
\left|T_n^{(k)}(t)\right| &\le (1 + o_{E'}(1)) \frac{8^k \pi^{2k}}{(2k-1)!!} n^{2k} \Omega(E'_{\mathbb{T}}, e^{ia})^{2k} \|T_n\|_{E'} \\
&\qquad\qquad \le (1 + o_E(1)) \frac{8^k \pi^{2k}}{(2k-1)!!} n^{2k} \Omega(E_{\mathbb{T}}, e^{ia})^{2k} \|T_n\|_E
\end{aligned}
$$

by letting $\varepsilon \to 0$ appropriately.

Fourth step. Now let $E \subset (-\pi, \pi)$ be a compact set which is regular (in the sense of Dirichlet problem). Obviously, the regularity of E and $E_{\mathbb{T}}$ are equivalent.

Consider the trigonometric polynomial $T_n Q_{n\varepsilon}$ of degree at most $n(1+\varepsilon)$ where $Q(.) = Q_{n\varepsilon}(.)$ is the fast decreasing polynomial with the following properties: its degree is at most $n\varepsilon$, $0 \le Q(.) \le 1$, $Q(t) \le \exp(-\delta_1 n\varepsilon)$ for some $\delta_1 > 0$ on $t \in [-\pi, a - 2\rho] \cup [a + 2\rho, \pi]$, $1 - \exp(-\delta_1 n\varepsilon) \le Q(t)$ on $t \in [a - \rho, a + \rho]$ and $Q(a) = 1$ (for existence, see Section 3.1).

Let $g_{E_{\mathbb{T}}}(\zeta, 0)$ and $g_{E_{\mathbb{T}}}(\zeta, \infty)$ be the Green functions of the domain $\overline{\mathbf{C}} \setminus E_{\mathbb{T}}$ with poles at the points 0 and ∞, respectively. The regularity of the set E (and $E_{\mathbb{T}}$ correspondingly) implies the continuity of $g_{E_{\mathbb{T}}}(\zeta, 0)$ and $g_{E_{\mathbb{T}}}(\zeta, \infty)$ at all points different from 0 and ∞, as well as the fact that these functions vanish at the points of $E_{\mathbb{T}}$. Therefore, for the $\delta_1 > 0$ there is a $d_1 > 0$, such that if $t \in \mathbf{R}$ and $\operatorname{dist}(t, E) \le d_1$, then

$$
g_{E_{\mathbb{T}}}(e^{it}, 0) < \frac{\delta_1^2}{2}. \tag{46}
$$

We choose m sufficiently so large that for the set E_m^+ the condition $\mathrm{dist}(t,E) \leq d_1$ for all $t \in E_m^+$ is satisfied.

If $t \in E$ then

$$|T_n(t)Q_{n\varepsilon}(t)| \leq \|T_n\|_E.$$

If we write

$$\begin{aligned} T_n(t) &= \sum_{j=0}^{n} (A_j \cos jt + B_j \sin jt) \\ &= \sum_{j=0}^{n} (\mathrm{Re}A_j \cos jt + \mathrm{Re}B_j \sin jt) + i \sum_{j=0}^{n} (\mathrm{Im}A_j \cos jt + \mathrm{Im}B_j \sin jt), \end{aligned}$$

we consider the algebraic polynomials

$$S_n^{(1)}(z) = \sum_{j=0}^{n} (\mathrm{Re}A_j - i\mathrm{Re}B_j)\, z^j, \quad S_n^{(2)}(z) = \sum_{j=0}^{n} (\mathrm{Im}A_j - i\mathrm{Im}B_j)\, z^j.$$

It is easy to verify that $T_n(t) = F(e^{it})$ for all complex t, where

$$F(z) := \frac{1}{2}\left[S_n^{(1)}(z) + \overline{S_n^{(1)}\left(\frac{1}{\overline{z}}\right)}\right] + \frac{i}{2}\left[S_n^{(2)}(z) + \overline{S_n^{(2)}\left(\frac{1}{\overline{z}}\right)}\right]$$

is a rational function. We note that $\|F\|_{E_{\mathbb{T}}} = \|T_n\|_E$ and apply an analog of the Bernstein-Walsh inequality (see e.g. [3], p. 64) to the rational function F on $E_{\mathbb{T}}$ and then use the fact that the domain $\overline{\mathbf{C}} \setminus E_{\mathbb{T}}$ is symmetric with respect to the unit circle. For simplicity, we put

$$g(z,w) = g_{\overline{\mathbf{C}} \setminus E_{\mathbb{T}}}(z,w)$$

for Green's function of $E_{\mathbb{T}}$. So, we have for $t \in \mathbf{R}$ that

$$\begin{aligned} |T_n(t)| = \left|F\left(e^{it}\right)\right| &\leq \|F\|_{E_{\mathbb{T}}} \exp\left(n\left(g(e^{it},0) + g(e^{it},\infty)\right)\right) \\ &= \|T_n\|_E \exp\left(2ng(e^{it},0)\right). \end{aligned}$$

Now if $t \in E_m^+ \setminus E$ then it follows from (21) and (46) that

$$\begin{aligned} |T_n(t)Q_{n\varepsilon}(t)| &\leq \|T_n\|_E \exp\left(2ng(e^{it},0)\right) \exp(-n\delta_1) \\ &\leq \|T_n\|_E \exp\left(n\delta_1^2 - n\delta_1\right) \leq \|T_n\|_E \end{aligned}$$

for sufficiently large n, and hence $\|T_nQ_{n\varepsilon}\|_{E_m^+} \leq \|T\|_E$.

For $t \in [a-\rho, a]$

$$\left|(T_nQ_{n\varepsilon})^{(k)}(t)\right| \geq \left|T^{(k)}(t)Q_{n\varepsilon}(t)\right| - \sum_{j=1}^{k} \binom{k}{j} \left|T_n^{(k-j)}(t)Q_{n\varepsilon}^{(j)}(t)\right|.$$

Here $1-e^{-n\delta_1} \le Q_{n\varepsilon}(t) \le 1$ and by (26)

$$\|Q_{n\varepsilon}^{(j)}\|_E \le C(n\varepsilon)^{2j}, \quad \|T_n^{(j)}\|_E \le Cn^{2j}\|T_n\|_E$$

with some constant C for all $j=1,2,\ldots,k$. Hence, if $t\in[a-\rho,a]$ we get from the previous step applied to the trigonometric polynomial $T_n(t)Q_{n\varepsilon}(t)$ on the set E_m^+ (which consists of finitely many intervals) that

$$\left|T_n^{(k)}(t)\right|\left(1-e^{-n\delta_1}\right) \le \left|(T_nQ_{n\varepsilon})^{(k)}(t)\right| + \sum_{j=1}^{k}\binom{k}{j}C^2\|T_n\|_E n^{2(k-j)}(n\varepsilon)^{2j}$$

$$\le (1+o(1))8^k\pi^{2k}\frac{1}{(2k-1)!!}\Omega\left((E_m^+)_{\mathbb{T}},e^{ia}\right)^{2k} \ (n(1+\varepsilon))^{2k}\,\|T_nQ_{n\varepsilon}\|_{E_m^+}+C_1\varepsilon^2n^{2k}\|T_n\|_E$$

$$\le \frac{n^{2k}}{(2k-1)!!}\|T_n\|_E\left((1+o(1))\,(1+\varepsilon)^{2k}8^k\pi^{2k}\Omega\left((E_m^+)_{\mathbb{T}},e^{ia}\right)^{2k}+C_1\varepsilon^2\right).$$

Since $\varepsilon>0$ and m are arbitrary, the inequality (28) follows from Lemma 1.

Fifth step. The regularity condition can be removed using the sets E_m^- and $(E_m^-)_{\mathbb{T}}$ from Ancona's theorem (interval condition (2) implies $[a-\rho,a]\subset E$, hence $\mathrm{cap}(E)>0$). Indeed,

$$\|T_n^{(k)}\|_{[a-\rho,a]} \le (1+o_m(1))\,\frac{n^{2k}}{(2k-1)!!}8^k\pi^{2k}\Omega\left((E_m^-)_{\mathbb{T}},e^{ia}\right)\|T_n\|_{E_m^-}$$

$$\le (1+o_m(1))\,\frac{n^{2k}}{(2k-1)!!}8^k\pi^{2k}\Omega\left((E_m^-)_{\mathbb{T}},e^{ia}\right)\|T_n\|_E$$

where $o_m(1)$ depends on E_m^- too.

It follows from Lemma 1 that $\Omega((E_m^-)_{\mathbb{T}},e^{ia})$ can be made arbitrary close to $\Omega(E_{\mathbb{T}},e^{ia})$ by choosing m large enough. Hence the inequality (28) holds in this case too.

Now we investigate the sharpness, that is, we are going to establish (29). As above, first we show it for the case when E is a union of finitely many intervals. We select a T-set as in Section 2.1 for which $\Omega(E'_{\mathbb{T}},e^{ia})$ is close to $\Omega(E_{\mathbb{T}},e^{ia})$, say $\Omega(E'_{\mathbb{T}},e^{ia})\ge\Omega(E_{\mathbb{T}},e^{ia})(1-\varepsilon)$ for some given $\varepsilon>0$.

By (32)

$$|U_N'(a)| = 8\pi^2N^2\Omega(E'_{\mathbb{T}},e^{ia})^2. \tag{47}$$

Now note that if $\mathcal{T}_l(x)=\cos(l\arccos(x))$ are classical Chebyshev polynomials, then $T_n(t):=\mathcal{T}_l(U_N(t))$ is a trigonometric polynomial of degree lN for which

$$E'=\{x|\mathcal{T}_l(U_N(x))\in[-1,1]\}.$$

Since

$$\left|\mathcal{T}_l^k(\pm1)\right| = \frac{l^2(l^2-1)\ldots(l^2-(k-1)^2)}{(2k-1)!!} =: C_{l,k}.$$

and (47) we get for $n = lN$ as before

$$\left|T_n^{(k)}(a)\right| = \left|(\mathcal{T}_l(U_N))^{(k)}(a)\right| = (1 \pm o(1))C_{l,k}N^{2k}8^k\pi^{2k}\Omega(E'_{\mathbb{T}}, e^{ia})^{2k},$$

and here, in view of (31),

$$C_{l,k}N^{2k}\Omega(E'_{\mathbb{T}}, e^{ia})^{2k} \geq (1 - o(1))\frac{l^{2k}}{(2k-1)!!}N^{2k}\Omega(E_{\mathbb{T}}, e^{ia})^{2k}(1-\varepsilon)^{2k}.$$

Since $E \subset E'$ we have

$$\|T_n\|_E \leq \|T_n\|_{E'} = \|\mathcal{T}_l\|_{[-1,1]} = 1,$$

and so from $n = lN$ we get

$$\left|T_n^{(k)}(a)\right| \geq (1 - o(1))^2(1-\varepsilon)^{2k}\frac{n^{2k}}{(2k-1)!!}8^k\pi^{2k}\Omega(E_{\mathbb{T}}, e^{ia})^{2k}\|T_n\|_E.$$

This is only for integers n of the form $n = lN$. For others just use $T_n(t) = \mathcal{T}_{[n/N]}(U_N(t)) + \delta\cos(nt)$ with $\delta > 0$ very small. Since here $\varepsilon = \varepsilon_N > 0$ is arbitrary, (29) follows if we let N tend to ∞ slowly and at the same time $U_N^{-1}[-1,1]$ approaches E, as $n \to \infty$ (in which case we have $\varepsilon_N \to 0$).

In the general case we consider the sets E_m^+ that are unions of finitely many intervals. Hence, we may use the last result for E_m^+, namely, there is a sequence of nonzero trigonometric polynomials $\{T_{m,n}\}_{n=1}^{\infty}$, $\deg(T_{m,n}) \leq n$, such that

$$\left|T_{m,n}^{(k)}(a)\right| \geq (1 - o_{E_m^+}(1))n^{2k}\Omega\left((E_m^+)_{\mathbb{T}}, e^{ia}\right)^{2k}\frac{8^k\pi^{2k}}{(2k-1)!!}\|T_{m,n}\|_{E_m^+},$$

where $o_{E_m^+}(1)$ depends on E_m^+ and it tends to 0 as $n \to \infty$ for any fixed m. Since $E \subset E_m^+$, we have $\|T_{m,n}\|_{E_m^+} \geq \|T_{m,n}\|_E$ and hence

$$\left|T_{m,n}^{(k)}(a)\right| \geq (1 - o_{E_m^+}(1))n^{2k}\Omega\left((E_m^+)_{\mathbb{T}}, e^{ia}\right)^{2k}\frac{8^k\pi^{2k}}{(2k-1)!!}\|T_{m,n}\|_E.$$

By Lemma 1 and choosing m sufficiently large, $\Omega\left((E_m^+)_{\mathbb{T}}, e^{ia}\right)$ can be made arbitrary close to $\Omega(E_{\mathbb{T}}, e^{ia})$. Therefore, (29) follows for $T_n := T_{m_n,n}$ if m_n goes slowly to infinity as $n \to \infty$. $\square$

Now if H denotes the shorter arc on $\mathbb{T}$ connecting the points $e^{i(a-\rho)}$ and e^{ia} then we have the following assertion.

Corollary 7. *Under the conditions mentioned above for any algebraic polynomial P_n with degree n, we have*

$$\left\|P_n^{(k)}\right\|_H \leq (1 + o(1))n^{2k}\Omega(E_{\mathbb{T}}, e^{ia})^{2k}\frac{2^k\pi^{2k}}{(2k-1)!!}\|P_n\|_{E_{\mathbb{T}}}. \tag{48}$$

This inequality is sharp, for there is a sequence of polynomials $P_n \not\equiv 0$, $n = 1, 2, \dots$, such that

$$\left|P_n^{(k)}(e^{ia})\right| \geq (1 - o(1))n^{2k}\Omega(E_{\mathbb{T}}, e^{ia})^{2k} \frac{2^k \pi^{2k}}{(2k-1)!!} \|P_n\|_{E_{\mathbb{T}}}. \tag{49}$$

The quantity $o(1)$ depends on E and k and tends to 0 as $n \to \infty$.

Proof. We may assume that n is even (because $(n+1)^2/n^2 = 1 + o(1)$). We consider the trigonometric polynomial $T_{n/2}(t) = e^{-itn/2} P_n\left(e^{it}\right)$. So, (48) follows now from applying Theorem 6 to $T_{n/2}$.

Concerning (49), existence of such polynomials, in view of the remark above, follows from existence of trigonometric polynomials T_n for which (29) holds.

□

4 Higher order Bernstein-type inequalities and their sharpness

Let $E \subset (-\pi, \pi)$ be a compact subset, and fix a point $z_0 = e^{it_0}$ which is in the one dimensional interior of $E_{\mathbb{T}}$. That is, $\{\exp(it) : t_0 - \delta < t < t_0 + \delta\} \subset E_{\mathbb{T}}$ for some small $\delta > 0$. Denote by $\partial/\partial \mathbf{n}_+$ and $\partial/\partial \mathbf{n}_-$ the outward and inward normal derivatives (w.r.t. the unit circle) correspondingly. Then (see [17], formulas (23) and (24) on p. 349)

$$\frac{1}{2}\left(1 + 2\pi\omega_{E_{\mathbb{T}}}\left(e^{it}\right)\right) = \frac{\partial g(e^{it}, \infty)}{\partial \mathbf{n}_+} = \max\left(\frac{\partial g(e^{it}, \infty)}{\partial \mathbf{n}_+}, \frac{\partial g(e^{it}, \infty)}{\partial \mathbf{n}_-}\right)$$

where $g(z, w) = g_{\overline{\mathbf{C}} \setminus E_{\mathbb{T}}}(z, w)$ is Green's function of $\overline{\mathbf{C}} \setminus E_{\mathbb{T}}$ and $\omega_{E_{\mathbb{T}}}(.)$ denotes the density of the equilibrium measure (w.r.t. arc length on the unit circle).

Now let us consider higher order Bernstein-type inequalities for trigonometric polynomials.

Theorem 8. *Let $E \subset (-\pi, \pi)$ be a compact set and k be a positive integer. Fix a closed interval $E_0 \subset \operatorname{Int} E$ (subset of the one dimensional interior of E). Then there exists $C = C(E, E_0, k) > 0$ such that for all trigonometric polynomial T_n with degree n, we have for $t \in E_0$*

$$\left|T_n^{(k)}(t)\right| \leq (1 + o(1))n^k \left(2\pi\omega_{E_{\mathbb{T}}}\left(e^{it}\right)\right)^k \|T_n\|_E. \tag{50}$$

where $o(1)$ is uniform in $t \in E_0$ and uniform among all trigonometric polynomials having degree at most n and tends to 0 as $n \to \infty$.

Proof. We prove the theorem by induction on k, the case $k = 1$ was done in [13, Theorem 4].

Let

$$V(t) = 2\pi\omega_{E_{\mathbb{T}}}\left(e^{it}\right).$$

Select a closed set $E_0^* \supset E_0$ such that E_0^* has no common endpoints either with E_0 or with E.

Consider any $\delta > 0$ such that the intersection of E with the δ-neighborhood of E_0 is still subset of of E_0^*, and set $f_{k,n,t_0}(t) := T_n^{(k)}(t)Q(t)$, where $Q(t) = Q_{n^{1/3}}(t)$ is a fast decreasing trigonometric polynomial from Theorem 3 for $t_0 \in E_0$ (α' and β' from Theorem 3 are chosen such a way that the interval $[\alpha', \beta']$ is in the δ-neighborhood of E_0).

By (23) and (26), for this f_{k,n,t_0} we have the upper bound

$$O(n^{2k}) \exp\left(-\delta_1 n^{1/3}\right) \|T_n\|_E = o(1)\|T_n\|_E$$

on E outside the δ-neighborhood of t_0 with $\delta_1 > 0$ (uniform in $t_0 \in E_0$).

In the δ-neighborhood of any $t_0 \in E_0$, by $\|Q\|_E \leq 1$ and by induction hypothesis applied to T_n and to E_0^*, we have

$$|f_{k,n,t_0}(t)| \leq (1 + o(1))n^k \|T_n\|_E V(t)^k \leq (1 + o(1))n^k (1 + \varepsilon)^k \|T_n\|_E V(t_0)^k,$$

where $\varepsilon \to 0$ as $\delta \to 0$. Here we used that by the continuity of $V(t)$, if $t_0 \in E_0$ and $|t - t_0| < \delta$, then $V(t) \leq (1+\varepsilon)V(t_0)$ with some ε that tends to 0 as $\delta \to 0$. Therefore, $f_{k,n,t_0}(t)$ is a trigonometric polynomial in t of degree at most $n+n^{1/3}$ for which

$$\|f_{k,n,t_0}\| \leq (1 + o(1))n^k \|T_n\|_E V(t_0)^k.$$

Upon applying Lukashov's theorem from [13, Theorem 4] to the trigonometric polynomial $f_{k,n,t_0}(t)$ we obtain

$$|f'_{k,n,t_0}(t_0)| \leq (1 + o(1))n^{k+1} \|T_n\|_E V(t_0)^{k+1}. \tag{51}$$

Since (recall that $Q(t_0) = 1$)

$$f'_{k,n,t_0}(t_0) = T_n^{(k+1)}(t_0) + T_n^{(k)}(t_0)(Q(t_0))',$$

and the second term on the right is at most $O(n^k)O(n^{2/3})\|T_n\|_E$ in modulus, by (26) and by the induction assumption, from (51) we get (50). It follows from the proof that the estimate is uniform in $t_0 \in E_0$.

□

Corollary 9. *Let $E \subset (-\pi, \pi)$ be again a compact set and k be a positive integer. Fix a closed interval $E_0 \subset \operatorname{Int} E$. Then there exists $C = C(E, E_0, k) > 0$ such that for all algebraic polynomial P_n with degree n, we have for $z = e^{it}$, $t \in E_0$*

$$\left|P_n^{(k)}(z)\right| \leq (1 + o(1)) \frac{n^k}{2^k} \left(1 + 2\pi\omega_{E_{\mathbb{T}}}(z)\right)^k \|P_n\|_{E_{\mathbb{T}}} \tag{52}$$

where $o(1)$ is uniform in $z = e^{it}$, $t \in E_0$ and independent of P_n, but it tends to 0 as $n \to \infty$.

Proof. As in the proof of Corollary 7, we may assume that n is even (because $(n+1)^2/n^2 = 1 + o(1)$) and consider the trigonometric polynomial $T_{n/2}(t) = e^{-itn/2} P_n\left(e^{it}\right)$. By Theorem 8, we get

$$\begin{aligned}(1+o(1))\frac{n^k}{2^k}\left(2\pi\omega_{E_{\mathbb{T}}}\left(e^{it}\right)\right)^k \|T_{n/2}\|_E &\geq |T_n^{(k)}(t)| \\ &\geq \left|\left(P_n\left(e^{it}\right)\right)^{(k)}\right| - \left|\sum_{j=0}^{k-1}\binom{k}{j}\left(P\left(e^{it}\right)\right)^{(j)}\left(e^{-itn/2}\right)^{(k-j)}\right|.\end{aligned}$$

It, together with Faà di Bruno's formula (1) and Theorem 8 yields that

$$\begin{aligned}\left|P_n^{(k)}(z)\right| &\leq (1+o(1))\frac{n^k}{2^k}\left((2\pi\omega_{E_{\mathbb{T}}}(z))^k + \sum_{j=0}^{k-1}\binom{k}{j}(2\pi\omega_{E_{\mathbb{T}}}(z))^j\right)\|P_n\|_{E_{\mathbb{T}}} \\ &\leq (1+o(1))\frac{n^k}{2^k}(1+2\pi\omega_{E_{\mathbb{T}}}(z))^k\|P_n\|_{E_{\mathbb{T}}}.\end{aligned}$$

□

Corollary 9 extends Theorem 1 of the paper [17] to higher derivatives of algebraic polynomials and the proof of sharpness is similar to the proof of [17], Theorem 2.

Theorem 10. *Under assumption of Corollary 9, inequality (52) is sharp, that is, there is a sequence of polynomials $P_n \not\equiv 0$, $n = 1, 2, \ldots$, such that*

$$\left|P_n^{(k)}(z)\right| \geq (1-o(1))\frac{n^k}{2^k}(1+2\pi\omega_{E_{\mathbb{T}}}(z))^k \|P_n\|_{E_{\mathbb{T}}}.$$

The quantity $o(1)$ depends on E and k and tends to 0 as $n \to \infty$.

Proof. We enclose $E_{\mathbb{T}}$ into a set G with the following properties:

- G is a finite union of disjoint C^2 smooth Jordan domains: there are finitely many disjoint C^2 Jordan curves $S_1, \ldots, S_m$ such that if G_j is the bounded connected components of $\overline{\mathbf{C}} \setminus S_j$, then $\overline{G} = \cup_{j=1}^m \overline{G}_j$,
- $E_{\mathbb{T}}$ is a boundary arc of the boundary ∂G,
- the component of G that contains z lies in the closed unit disk,
- every point of G is of distance $\leq \eta$ from a point of $E_{\mathbb{T}}$, where η is a given positive number.

Then the boundary $\Gamma = \partial G = \cup_{j=1}^m S_j$ is a family of disjoint Jordan curves. Furthermore, let $\mathbf{n}_+ = z$ be the normal at z to Γ pointed to the interior of $\Omega = \overline{\mathbf{C}} \setminus G$.

If $\varepsilon > 0$ is given, then for sufficiently small η we have (see e.g. [15], pp. 350-351

$$\frac{\partial g_\Omega(z,\infty)}{\partial \mathbf{n}_+} \geq (1-\varepsilon)\frac{\partial g_{\overline{\mathbf{C}}\setminus E_{\mathbb{T}}}(z,\infty)}{\partial \mathbf{n}_+}. \tag{53}$$

By the sharp form of the Hilbert lemniscate theorem [15], Theorem 1.2, there is a Jordan curve σ such that

- σ contains Γ in its interior except for the point z, where the two curves touch each other,
- σ is a lemniscate, i.e. $\sigma = \{\zeta : |V_N(\zeta)| = 1\}$ for some algebraic polynomial V_N of degree N, and
- $$\frac{\partial g_{\overline{\mathbf{C}}\setminus\sigma}(z,\infty)}{\partial \mathbf{n}_+} \geq (1-\varepsilon)\frac{\partial g_\Omega(z,\infty)}{\partial \mathbf{n}_+}. \tag{54}$$

We may assume that $V_N'(z) > 0$. The Green's function of the outer domain of σ is $\frac{1}{N}\log|V_N(.)|$, and its normal derivative is

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus\sigma}(z,\infty)}{\partial \mathbf{n}_+} = \frac{1}{N}|V_N'(z)| = \frac{1}{N}V_N'(z).$$

Consider now, for all large n, the polynomials $P_n(.) = V_N(.)^{[n/N]}$. This is a polynomial of degree at most n, its supremum norm on σ is 1, and by Faà di Bruno formula (1), it can be shown that (see also [8], subsection 10.2)

$$\left|P_n^{(k)}(z)\right| = n^k\left(\frac{\partial g_{\overline{\mathbf{C}}\setminus\sigma}(z,\infty)}{\partial \mathbf{n}_+}\right)^k + O(n^{k-1}).$$

Thus, in view of (53) and (54), we may continue

$$\left|P_n^{(k)}(z)\right| \geq (1-\varepsilon)^{2k} n^k\left(\frac{\partial g_{\overline{\mathbf{C}}\setminus E_{\mathbb{T}}}(z,\infty)}{\partial \mathbf{n}_+}\right)^k + O(n^{k-1}).$$

Note also that $\|P_n\|_{E_{\mathbb{T}}} \leq \|P_n\|_\sigma = 1$ by the maximum principle. $\square$

Corollary 11. *Under assumption of Theorem 8, inequality (50) is sharp, for there is a sequence of trigonometric polynomials $T_n \not\equiv 0$, $n = 1, 2, \ldots$, such that*

$$\left|T_n^{(k)}(t)\right| \geq (1-o(1))n^k\left(2\pi\omega_{E_{\mathbb{T}}}\left(e^{it}\right)\right)^k \|T_n\|_E\,.$$

where $o(1)$ depends on E and k and tends to 0 as $n \to \infty$.

Proof. Existence of such trigonometric polynomials T_n follows immediately from the existence of corresponding (in the sense of the proof of Corollary 9) algebraic polynomials P_{2n} from Corollary 9. $\square$

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