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Qualitative Theory of Dynamical Systems

Periodic Solutions and Hydra Effect for Delay Differential Equations with Nonincreasing Feedback

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Abstract We consider a delay differential equation modeling production and destruction, and prove the presence of the paradoxial hydra effect. Namely, for the equation $\dot{y}(t) = -\mu y(t) + f(y(t-1))$ with $\mu > 0$ and nonincreasing $f : \mathbb{R} \to (0, \infty)$, it is shown that the mean value of certain solutions can be increased by increasing the value of the (destruction) parameter μ. The nonlinearity f in the equation is a step function or a smooth function close to a step function. This particular form of f allows us to construct periodic solutions, and to evaluate the mean values of the periodic solutions. Our result explains how the global form of the nonlinearity f (the production term) induces the appearance of the hydra effect.

Keywords Delay differential equation · Periodic solution · Mean value · Hydra effect · Nonincreasing feedback

Mathematics Subject Classification 34K13 · 92D25 · 34K27

1 Introduction

Many processes can be characterized by a time dependent quantity $y(t)$, and the rate of change dy/dt can be given as a balance between the production rate p and the destruction rate d of y, that is, $dy/dt = p - d$. In general, through feedback and other interactive mechanisms, the production p and the destruction d at time t can

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depend on the quantity y in a complicated manner, see e.g., [5]. Here we assume that the model equation is a delay differential equation of the form

$$\dot{y}(t) = -\mu y(t) + f(y(t-h)) \tag{1.1}$$

with constants $\mu > 0$, $h > 0$, and a nonincreasing feedback function $f : \mathbb{R} \to (0, \infty)$. Equation (1.1) models the time change of the nonnegative quantity y such that y decays with a rate proportional to y at the present time t, and y is produced at time t with a rate dependent on the value of y at h time earlier. That is, $\mu y(t)$ describes the destruction, while $f(y(t-h))$ describes the production. Although Eq. (1.1) is a very simple delay differential equation, it has been successfully applied to a wide variety of nonlinear phenomena to demonstrate very complex dynamic behaviours [3,17]. For example, it appears in population dynamics, neural networks, physiology, economics.

In order to understand the process modeled by Eq. (1.1), it is important to know how the dynamics of (1.1) changes as the parameters μ, h, f change. This is a highly nontrivial bifurcation problem even for the simple looking Eq. (1.1). If μ and f are fixed, then, under mild conditons on f, it has been shown that with the increase of the delay parameter h there is a sequence of Hopf bifurcations, and for monotone (increasing or decreasing) f, the global dynamics is fairly well described [3,6,7,11, 12]. In this paper h and f will be fixed and the destruction (mortality) parameter μ will be increased, and we are interested in how the relevant quantitiy y changes. In [8] it has been proved that bubbles can be created in such a way: periodic orbits can arise through a Hopf bifurcation, and they can disappear in another Hopf bifurcation point.

Recently, several experimental and theoretical studies observed a counter-intuitive phenomenon, the so-called hydra effect [1,2], in different mathematical models for population sizes: the size of the population can be increased by increasing the mortality rate. This phenomenon has been named after the nine-headed beast in Greek mythology that grows two more heads for each one cut off. We refer to the papers [1,15] and references therein for a review of different models and mechanisms predicting positive mortality effects. Schröder et al. [15] also lists experimental results, and shows what types of mortality effects occur in natural populations. See also [19] for another hydra type effect. Understanding the positive mortality effects is a challenging problem, and it is very important for a wide range of applications (see [1,15]) not only in population models, but also in more general models of destruction and production.

The objective of this paper is to show analytically that hydra effect can appear in Eq. (1.1). The feedback function f will be either a certain step function or a continuous function close to a step function. By rescaling the time it can always be assumed that $h = 1$. Therefore we study the equation

$$\dot{y}(t) = -\mu y(t) + f(y(t-1)) \tag{1.2}$$

with a constant $\mu > 0$ and a monotone nonincreasing feedback function $f : \mathbb{R} \to (0, \infty)$ which is either continuous or a step function.

It is well known (see, e.g., [3,4,17]) that for any $\varphi \in C := C([-1, 0], \mathbb{R})$, Eq. (1.2) has a unique solution $y : [-1, \infty) \to \mathbb{R}$ with initial segment $y|_{[-1,0]} = \varphi$, i.e.,

y is continuous on $[-1, \infty)$, y is absolutely continuous on $(0, \infty)$, and (1.2) holds almost everywhere on $(0, \infty)$. This unique solution can be obtained by the method of steps. If we want to emphasize the dependence of the solutions of Eq. (1.2) on φ, f and μ, we write $y(\cdot, \varphi, f, \mu)$ instead of $y(\cdot)$. The same $y(\cdot, \varphi, f, \mu)$ will be used to denote solutions defined on the whole real line $\mathbb{R}$. In applications $y(t)$ usually denotes a nonnegative quantity, so it is natural to expect the following nonnegativity property: if $\varphi \in C_+ := C([-1, 0], [0, \infty))$ then $y(t, \varphi, f, \mu) \geq 0$ for all $t \geq 0$. It is elementary to verify this nonnegativity since the range of f is in $(0, \infty)$.

For a continuous function $u : I \to \mathbb{R}$, given on an interval $I \supset [0, \infty)$, let the mean value $MV[u]$ be defined as

$$MV[u] = \lim_{t \to \infty} \frac{1}{t} \int_0^t u(s) \, \mathrm{d}s$$

provided that the limit exists. If $u : \mathbb{R} \to \mathbb{R}$ is continuous and ω-periodic with $\omega > 0$, then $MV[u]$ obviously exists and

$$MV[u] = \frac{1}{\omega} \int_0^\omega u(\tau) \mathrm{d}\tau.$$

It is also clear that if $u : \mathbb{R} \to \mathbb{R}$ is continuous and ω-periodic, and $\tilde{u} : I \to \mathbb{R}$ is continuous with $\lim_{t\to\infty}[\tilde{u}(t + t_0) - u(t)] = 0$ for some $t_0 \geq 0$, then $MV[\tilde{u}]$ also exists, and $MV[\tilde{u}] = MV[u]$.

We say that hydra effect occurs in Eq. (1.2) if there are parameters $\mu_2 > \mu_1 > 0$ and there exists an initial function $\varphi \in C_+$ such that

$$MV[y(\cdot, \varphi, f, \mu_2)] > MV[y(\cdot, \varphi, f, \mu_1)].$$

This definition is in correspondence with the definition given by Sieber and Hilker in [16] for ordinary differential equations, and with the definition used by Liz and Ruiz-Herrera in [13,14] for difference equations.

In order to prove hydra effect for Eq. (1.2) the main difficulty is to find suitable nonlinearities f. This is a crucial step because we have to be able to evaluate the mean values of certain solutions. We overcome this difficulty so that we look for nonlinearities in the set of step functions parametrized by three parameters. Let $a > 0$, $b \in [0, 1)$, $c \in (0, 1)$, and define the step function $\mathrm{Step}(a, b, c) \colon \mathbb{R} \to (0, \infty)$ by

$$\mathrm{Step}(a, b, c)\,(y) = \begin{cases} 1 + a & \text{if } y < 1 - b, \\ 1 & \text{if } 1 - b \leq y \leq 1\ , \\ 1 - c & \text{if } y > 1 \end{cases} \tag{1.3}$$

(see Fig. 1).

We consider Eq. (1.2) with

$$f = \mathrm{Step}(a, b, c) \quad \text{and} \quad \mu \in \left[1, \frac{1}{1-b}\right].$$

Fig. 1 The plot of the step function Step(a, b, c)

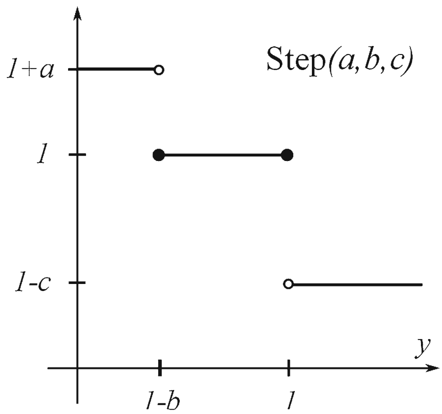
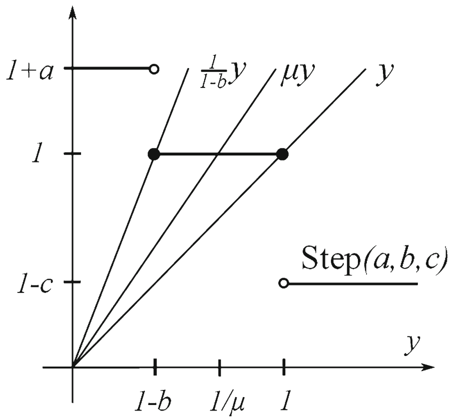


Fig. 2 The equlibrium as a function of $\mu \in \left[1, \frac{1}{1-b}\right]$



Note that the inequality $1 \leq \mu \leq 1/(1-b)$ for μ guarantees that $1/\mu$ is the unique equilibrium. See Fig. 2.

In Sect. 2, for suitable fixed parameters a, b and c, we explicitly construct periodic solutions $p(a, b, c, \mu) : \mathbb{R} \to \mathbb{R}$ of Eq. (1.2) with $f = \text{Step}(a, b, c)$ for all values of μ from the interval $[1, 1/(1-b)]$. The periodic solutions $p(a, b, c, \mu)$ are piecewise elementary functions because of the step function nonlinearity. This makes it possible to evaluate the mean values

$$MV[p(a, b, c, \mu)].$$

Another cruicial point is that for all initial functions

$$\psi \in \mathcal{A} := \{\varphi \in C : \varphi(\theta) \geq 1 \quad \text{for all } \theta \in [-1, 0]\}$$

the solution $y(\cdot, \psi, f, \mu)$ is eventually periodic, with $f = \text{Step}(a, b, c)$, $\mu \in [1, 1/(1-b)]$, and a, b, c fixed as above. More precisely, there exists $t_* = t_*(a, b, c, \mu, \psi) \geq 0$ such that

$$y(t + t_*, \psi, f, \mu) = p(a, b, c, \mu)(t) \quad \text{for all } t \geq 0.$$

Then for all $\psi \in \mathcal{A}$ we know the mean values since

$$MV[y(\cdot, \psi, f, \mu)] = MV[p(a, b, c, \mu)].$$

Section 3 verifies that it is possible to choose parameters $a > 0$, $b \in (0, 1)$, $c \in (0, 1)$ such that, with $f = \text{Step}(a, b, c)$, the inequality

$$MV\left[p\left(a, b, c, \frac{1}{1-b}\right)\right] > MV\left[p\left(a, b, c, 1\right)\right]$$

holds for the mean values. Then it easily follows that, for all $\psi \in \mathcal{A}$, we have

$$MV\left[y\left(\cdot, \psi, f, \frac{1}{1-b}\right)\right] > MV\left[y\left(\cdot, \psi, f, 1\right)\right].$$

This means that hydra effect occurs in Eq. (1.2) with $f = \text{Step}(a, b, c)$, $\mu_1 = 1$, $\mu_2 = 1/(1-b)$, and for all initial functions $\psi \in \mathcal{A}$.

Section 4 shows that hydra effect can be obtained for Eq. (1.2) with certain Lipschitz continuous feedback functions f as well. The nonlinearity f is assumed to be sufficiently close (in a precise sense given in Sect. 4) to the step function $\text{Step}(a, b, c)$. The proof of this statement uses the strong attractivity property of the periodic orbits obtained for the step function nonlinearity. Then a perturbation technique of Walther [21] is applied, see also [9,10,20]. Since the argument of [21] can be closely followed, here only the main steps are given.

In Sect. 5 we give a heuristic justification why our construction for hydra effect works. In fact, this reasoning was our starting point, and Sects. 2–4 show analytically its correctness. Another example, where hydra effect does not occur, is also given to support the heuristic explanation of the mechanism for hydra effect.

As far as we know there are not many analytical results for hydra effects in delay differential equations. Teismann et al. [18] have proved hydra effect in Nicholson's well-known blowflies equation ((1.2) with $f(x) = rxe^{-\nu x}$) for the bifurcated periodic orbits in a small neighborhood of a Hopf bifurcation point (μ_0, x_0). In [18] the local property of the nonlinearity f near x_0 guarantees the hydra effect. Our construction is of global nature in the sense that we allow μ to change on a large scale, i.e., not only in a small neighborhood of a critical point. Moreover, the considered periodic orbits are not in a small neighborhood of the equilibrium point.

2 Periodic Solutions for Step Feedback Functions

The first step is to construct periodic solutions for (1.2) with initial segments in the set $\mathcal{A}$.

Theorem 2.1 *For all $a > 0$ and $c \in (0, 1)$ fixed, there exists $b_0 = b_0(a, c) \in (0, 1)$ such that for all $b \in [0, b_0)$, Eq. (1.2), with $f = \text{Step}(a, b, c)$ and $\mu \in [1, 1/(1-b)]$, admits a periodic solution $p(a, b, c, \mu) : \mathbb{R} \to \mathbb{R}$ with the following properties.*

(i) *The initial segment* $p(a,b,c,\mu)|_{[-1,0]}$ *of* $p(a,b,c,\mu)$ *belongs to* $\mathcal{A}$.
(ii) *If* $\psi \in \mathcal{A}$ *and* $y(\cdot,\psi,f,\mu)$ *is the solution of* (1.2) *with* $f = \mathrm{Step}(a,b,c)$*, then there exists* $t_* = t_*(a,b,c,\mu,\psi) \geq 0$ *such that*

$$y(t+t_*,\psi,f,\mu) = p(a,b,c,\mu)(t) \quad \textit{for all } t \geq 0.$$

In the sequel, for the parameters a, b, c, μ, we always assume that $a > 0$, $b \in [0,1)$, $c \in (0,1)$, and $\mu \in [1, 1/(1-b)]$. We prove the theorem by explicitly constructing the periodic solutions.

In order to simplify the forthcoming computations, we apply the transformation $y = x + 1/\mu$ and study the equivalent equation

$$\dot{x}(t) = -\mu x(t) + g(x(t-1)) \tag{2.1}$$

with $g = \mathrm{Step}_\mu(a,b,c)$, where

$$\begin{aligned}\mathrm{Step}_\mu(a,b,c)(x) &= \mathrm{Step}(a,b,c)\left(x+\frac{1}{\mu}\right) - 1\\ &= \begin{cases} a & \text{if } x < 1-\frac{1}{\mu}-b,\\ 0 & \text{if } 1-\frac{1}{\mu}-b \leq x \leq 1-\frac{1}{\mu},\\ -c & \text{if } x > 1-\frac{1}{\mu},\end{cases}\end{aligned}$$

see Fig. 3. With this transformation we shifted the unique equilibrium $1/\mu$ to the origin.

The following observation is of key importance. Let $J_{-c} = (1-1/\mu, \infty)$, $J_0 = [1-1/\mu-b, 1-1/\mu]$ and $J_a = (-\infty, 1-1/\mu-b)$ denote the level sets of $\mathrm{Step}_\mu(a,b,c)$. Suppose that $0 \leq t_0 < t_1$, and $x\colon [-1,\infty) \to \mathbb{R}$ is a solution of equation (2.1) with $g = \mathrm{Step}_\mu(a,b,c)$. If for some $i \in \{-c, 0, a\}$, $x(t-1) \in J_i$ for all $t \in (t_0, t_1)$, then Eq. (2.1) reduces to the ordinary differential equation

$$\dot{x}(t) = -\mu x(t) + i \tag{2.2}$$

on the interval (t_0, t_1). By the continuity of x, it follows that

$$x(t) = \Psi_i(t-t_0, x(t_0)) \quad \text{for all } t \in [t_0, t_1], \tag{2.3}$$

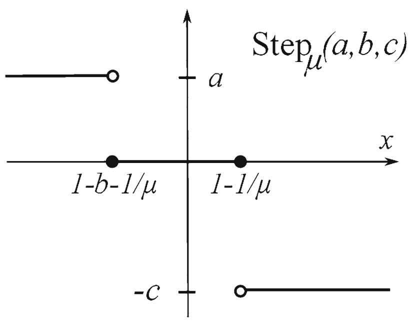


Fig. 3 The plot of $\mathrm{Step}_\mu(a,b,c)$

where Ψ_i denotes the flow generated by (2.2):

$$\Psi_i : \mathbb{R} \times \mathbb{R} \ni \left(s, x^*\right) \mapsto \frac{i}{\mu} + \left(x^* - \frac{i}{\mu}\right) e^{-\mu s} \in \mathbb{R}.$$

In this case we say that x is of type (i) on $[t_0, t_1]$.

We begin with a proposition stating that the solutions return infinitely often to the interval $[1 - 1/\mu - b, 1 - 1/\mu]$.

Proposition 2.2 *Assume that* $x\colon [-1, \infty) \to \mathbb{R}$ *is a solution of* (2.1) *with* $g = \mathrm{Step}_\mu(a, b, c)$. *Then there exists a sequence* $(t_n)_{n=0}^{\infty}$ *in* $[0, \infty)$ *so that* $t_n \to \infty$ *as* $n \to \infty$ *and*

$$x(t_n) \in \left[1 - \frac{1}{\mu} - b, 1 - \frac{1}{\mu}\right] \quad \textit{for all integers } n \geq 0.$$

Proof If the statement is not true then either there is $T_0 \geq 0$ so that $x(t) > 1 - 1/\mu$ for all $t > T_0$, or there is $T_1 \geq 0$ such that $x(t) < 1 - 1/\mu - b$ for all $t \geq T_1$. If $x(t) > 1 - 1/\mu$ for all $t > T_0$ with some $T_0 \geq 0$, then x is of type $(-c)$ on $[T_0 + 1, \infty)$, i.e.,

$$\begin{aligned} x(t) &= \Psi_{-c}\left(t - (T_0 + 1), x(T_0 + 1)\right) \\ &= \frac{-c}{\mu} + \left(x(T_0 + 1) + \frac{c}{\mu}\right) e^{\mu(T_0 + 1 - t)} \quad \text{for } t \geq T_0 + 1. \end{aligned}$$

It follows that $\lim_{t\to\infty} x(t) = -c/\mu < 0$, which contradicts the initial assumption. The nonexistence of T_1 can be proved in an analogous way. □

Introduce the closed subset

$$\mathcal{B}_\mu = \left\{\varphi \in C : \varphi(\theta) \geq 1 - \frac{1}{\mu} \quad \text{for all } \theta \in [-1, 0),\ \varphi(0) = 1 - \frac{1}{\mu}\right\} \tag{2.4}$$

of C for all $\mu \in [1, 1/(1 - b)]$.

Consider an arbitrary initial function $\varphi \in \mathcal{B}_\mu$ and the corresponding solution x of (2.1) with $g = \mathrm{Step}_\mu(a, b, c)$ and $x|_{[-1,0]} = \varphi$. In the following we calculate the values of x for certain choices of parameters a, b, c and μ. Thereby we show that with a suitable $\varphi \in \mathcal{B}_\mu$, we get a periodic solution.

As $\varphi(\theta) \geq 1 - 1/\mu$, $-1 \leq \theta \leq 0$,

$$x(t) = \Psi_{-c}\left(t, 1 - \frac{1}{\mu}\right) = \frac{-c}{\mu} + \left(1 - \frac{1 - c}{\mu}\right) e^{-\mu t} \quad \text{for all } t \in [0, 1]. \tag{2.5}$$

We will choose the parameters a, b, c and μ such that $x(1) < 1 - 1/\mu - b$ holds, that is,

$$\frac{-c}{\mu} + \left(1 - \frac{1 - c}{\mu}\right) e^{-\mu} < 1 - \frac{1}{\mu} - b \tag{2.6}$$

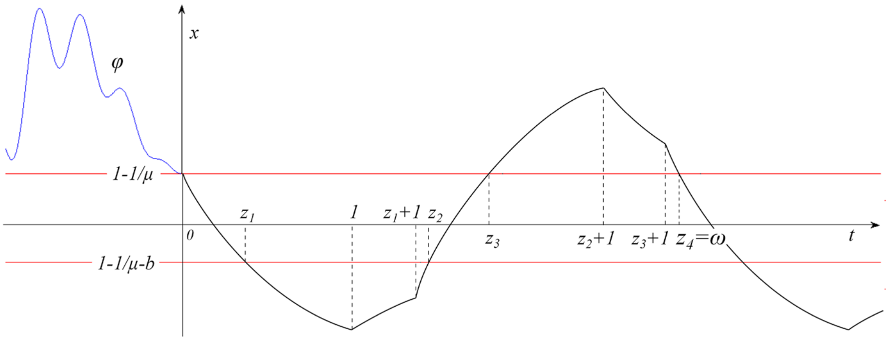


Fig. 4 The solution x of (2.1)

is satisfied. Let

$$U_1 = \left\{ (a, b, c, \mu) : a > 0, b \in [0, 1), c \in (0, 1), \mu \in \left[1, \frac{1}{1-b}\right] \text{ and (2.6) holds} \right\}.$$

For $(a, b, c, \mu) \in U_1$, there exists a time $z_1 \in [0, 1)$ such that $x(z_1) = 1 - 1/\mu - b$, see Fig. 4. As $1 - (1-c)/\mu > 0$, (2.5) shows that x is strictly decreasing on $[0, 1]$, thus z_1 is unique. It comes from (2.5) that

$$z_1 = \frac{1}{\mu} \ln \frac{1 - \frac{1-c}{\mu}}{1 - \frac{1-c}{\mu} - b}. \tag{2.7}$$

Since x is strictly decreasing on $[0, 1]$, $x(t) \in (1 - 1/\mu - b, 1 - 1/\mu)$ for $t \in (0, z_1)$. It follows that x is of type (0) on $[1, z_1 + 1]$, that is,

$$\begin{aligned} x(t) &= \Psi_0(t-1, x(1)) = x(1) e^{\mu(1-t)} \\ &= \frac{-c}{\mu} e^{\mu(1-t)} + \left(1 - \frac{1-c}{\mu}\right) e^{-\mu t} \quad \text{for } t \in [1, z_1 + 1]. \end{aligned} \tag{2.8}$$

Note that (2.5) has been used to get the last line of the above equality. Since $x(1) < 0$, (2.8) shows that x is strictly increasing on $[1, z_1 + 1]$.

For $t = z_1 + 1$, formulas (2.7) and (2.8) give that

$$x(z_1 + 1) = \left[1 - \frac{1-c}{\mu} - b\right] \left[e^{-\mu} - \frac{\frac{c}{\mu}}{1 - \frac{1-c}{\mu}}\right]. \tag{2.9}$$

We consider the situation when $x(z_1 + 1) < 1 - 1/\mu - b$, i.e.,

$$\left[1 - \frac{1-c}{\mu} - b\right] \left[e^{-\mu} - \frac{\frac{c}{\mu}}{1 - \frac{1-c}{\mu}}\right] < 1 - \frac{1}{\mu} - b. \tag{2.10}$$

So let

$$U_2 = \{(a, b, c, \mu) \in U_1 : \text{ (2.10) is satisfied}\}.$$

For all $(a, b, c, \mu) \in U_2$, Proposition 2.2 guarantees the existence of a minimal time $z_2 \in (z_1 + 1, \infty)$ such that $x(z_2) = 1 - 1/\mu - b$, see Fig. 4. Then observe that $x(t) < 1 - 1/\mu - b$ for all $t \in (z_1, z_2)$ because $x(z_1) = 1 - 1/\mu - b$, x strictly decreases on $[z_1, 1]$, x strictly increases on $[1, z_1 + 1]$, $x(z_1 + 1) < 1 - 1/\mu - b$, and $z_2 > z_1 + 1$ is minimal with $x(z_2) = 1 - 1/\mu - b$. Thus x is of type (a) on $[z_1 + 1, z_2 + 1]$, and applying (2.9), we deduce that

$$\begin{aligned} x(t) &= \Psi_a(t - z_1 - 1, x(z_1 + 1)) \\ &= \frac{a}{\mu} + \left(x(z_1 + 1) - \frac{a}{\mu}\right) e^{\mu(z_1+1-t)} \\ &= \frac{a}{\mu} + \left[\left(1 - \frac{1-c}{\mu} - b\right)\left(e^{-\mu} - \frac{\frac{c}{\mu}}{1 - \frac{1-c}{\mu}}\right) - \frac{a}{\mu}\right] e^{\mu(z_1+1-t)} \end{aligned} \tag{2.11}$$

for all $t \in [z_1 + 1, z_2 + 1]$. From $x(z_2) = 1 - 1/\mu - b$ we conclude that

$$z_2 = z_1 + 1 + \frac{1}{\mu} \ln \frac{\left(1 - \frac{1-c}{\mu} - b\right)\left(e^{-\mu} - \frac{\frac{c}{\mu}}{1-\frac{1-c}{\mu}}\right) - \frac{a}{\mu}}{1 - \frac{a+1}{\mu} - b}. \tag{2.12}$$

Then it is easy to see that

$$x(t) = \Psi_a\left(t - z_2, 1 - \frac{1}{\mu} - b\right) = \frac{a}{\mu} + \left(1 - \frac{a+1}{\mu} - b\right) e^{\mu(z_2-t)}. \tag{2.13}$$

holds for $t \in [z_1 + 1, z_2 + 1]$. As $1/\mu \geq 1 - b$, we have

$$1 - \frac{a+1}{\mu} - b \leq -a(1-b) < 0,$$

which means that x is strictly increasing on $[z_1 + 1, z_2 + 1]$.

Consider the inequality

$$\frac{a}{\mu} + \left(1 - \frac{a+1}{\mu} - b\right) e^{-\mu} > 1 - \frac{1}{\mu} \tag{2.14}$$

and the subset

$$U_3 = \{(a, b, c, \mu) \in U_2 : \text{ (2.14) holds}\}$$

of U_2. For $(a, b, c, \mu) \in U_3$, we have $x(z_2 + 1) > 1 - 1/\mu$. Let $z_3 \in (z_2, z_2 + 1)$ be a time for which $x(z_3) = 1 - 1/\mu$, see Fig. 4. Since x is strictly increasing on $[z_2, z_2+1]$, z_3 is unique, and $x(t) \in (1 - 1/\mu - b, 1 - 1/\mu)$ for $t \in (z_2, z_3)$. Formula (2.13) yields that

$$z_3 = z_2 + \frac{1}{\mu} \ln \frac{1 - \frac{a+1}{\mu} - b}{1 - \frac{a+1}{\mu}}. \tag{2.15}$$

The solution x is of type (0) on $[z_2+1, z_3+1]$, that is

$$\begin{aligned} x(t) &= \Psi_0 (t - z_2 - 1, x(z_2+1)) \\ &= x(z_2+1)\, e^{\mu(z_2+1-t)} \\ &= \frac{a}{\mu} e^{\mu(z_2+1-t)} + \left(1 - \frac{a+1}{\mu} - b\right) e^{\mu(z_2-t)} \quad \text{for } t \in [z_2+1, z_3+1]. \end{aligned} \tag{2.16}$$

In particular, the last two results imply that

$$x(z_3+1) = \left[1 - \frac{a+1}{\mu}\right]\left[e^{-\mu} + \frac{\frac{a}{\mu}}{1 - \frac{a+1}{\mu} - b}\right]. \tag{2.17}$$

We see that x is strictly decreasing on $[z_2+1, z_3+1]$ as $x(z_2+1) > 0$.

At last, consider the subset of U_3 for which $x(z_3+1) > 1 - 1/\mu$ holds, that is,

$$\left[1 - \frac{a+1}{\mu}\right]\left[e^{-\mu} + \frac{\frac{a}{\mu}}{1 - \frac{a+1}{\mu} - b}\right] > 1 - \frac{1}{\mu}. \tag{2.18}$$

So define

$$U_4 = \{(a, b, c, \mu) \in U_3 : \text{(2.18) is satisfied}\}.$$

Proposition 2.2 implies that, for all $(a, b, c, \mu) \in U_4$, a minimal $z_4 > z_3 + 1$ can be given with $x(z_4) = 1 - 1/\mu$. We have $x(t) > 1 - 1/\mu$ for all $t \in (z_3, z_4)$, hence x is of type $(-c)$ on $[z_3+1, z_4+1]$:

$$\begin{aligned} x(t) &= \Psi_{-c} (t - z_3 - 1, x(z_3+1)) \\ &= -\frac{c}{\mu} + \left(x(z_3+1) + \frac{c}{\mu}\right) e^{\mu(z_3+1-t)} \\ &= -\frac{c}{\mu} + \left[\left(1 - \frac{a+1}{\mu}\right)\left(e^{-\mu} + \frac{\frac{a}{\mu}}{1 - \frac{a+1}{\mu} - b}\right) + \frac{c}{\mu}\right] e^{\mu(z_3+1-t)} \end{aligned} \tag{2.19}$$

for $t \in [z_3+1, z_4+1]$. Note that we used (2.17) in the last line. The equation $x(z_4) = 1 - 1/\mu$ implies that

$$z_4 = z_3 + 1 + \frac{1}{\mu} \ln \frac{\left(1 - \frac{a+1}{\mu}\right)\left(e^{-\mu} + \frac{\frac{a}{\mu}}{1 - \frac{a+1}{\mu} - b}\right) + \frac{c}{\mu}}{1 - \frac{1-c}{\mu}}. \tag{2.20}$$

Then it is clear that

$$\begin{aligned} x(t) &= \Psi_{-c}(t - z_4, x(z_4)) \\ &= -\frac{c}{\mu} + \left(1 - \frac{1-c}{\mu}\right) e^{\mu(z_4 - t)} \quad \text{for all } t \in [z_3 + 1, z_4 + 1]. \end{aligned} \tag{2.21}$$

We have already noted that $x(t) > 1 - 1/\mu$ for all $t \in (z_3, z_4)$. As $z_4 - z_3 > (z_3 + 1) - z_3 = 1$ and $x(z_4) = 1 - 1/\mu$, obviously $x(z_4 + \cdot)|_{[-1,0]} \in \mathcal{B}_\mu$.

Observe that the above calculations are valid for all initial functions $\varphi \in \mathcal{B}_\mu$, independently of the concrete choice of φ.

The following proposition confirms that solutions of the above form exist, i.e., the set U_4 is nonempty.

Proposition 2.3 *For all $a > 0$ and $c \in (0, 1)$ fixed, there exists $b_0 = b_0(a, c) \in (0, 1)$ such that $(a, b, c, \mu) \in U_4$ for all $b \in [0, b_0)$ and $\mu \in [1, 1/(1 - b)]$.*

Proof Assume that $a > 0$ and $c \in (0, 1)$ are fixed. Notice that for all $b \in [0, 1)$ and $\mu \in [1, 1/(1-b)]$, the 4-tuple (a, b, c, μ) is in U_4 if and only if the inequalities (2.6), (2.10), (2.14) and (2.18) hold. For all $b \in [0, 1)$ and $\mu \in [1, 1/(1 - b)]$, we have $1 - (a + 1)/\mu - b \leq -a(1 - b) < 0$ and $1 - (1 - c)/\mu \geq c > 0$. It follows that all functions appearing in the inequalities (2.6), (2.10), (2.14) and (2.18) are continuous in a, b, c, μ.

Observe that at the point $(a, 0, c, 1)$, the inequalities (2.6), (2.10), (2.14), (2.18) reduce to the inequalities

$$c\left(1 - e^{-1}\right) > 0 \quad \text{and} \quad a\left(1 - e^{-1}\right) > 0,$$

which are obviously satisfied. By continuity, for all fixed $a > 0$ and $c \in (0, 1)$ we can find a $b_0 = b_0(a, c) \in (0, 1)$ such that all inequalities are still valid for all $b \in [0, b_0)$ and for all $\mu \in [1, 1/(1 - b)] \subset [1, 1/(1 - b_0)]$. □

Proof of Theorem 2.1 Let $(a, b, c, \mu) \in U_4$, and fix $\varphi \in \mathcal{B}_\mu$. Consider the solution x of (2.1) with $g = \mathrm{Step}_\mu(a, b, c)$ and inital segment φ. Solution x restricted to $[0, z_4 + 1]$ is given by formulas (2.5), (2.8), (2.13), (2.16) and (2.19). Let us introduce the notation $\omega = z_4$ where z_4 is defined in (2.20). Recall that $x(z_4) = 1 - 1/\mu$ and $x(\omega + \cdot)|_{[-1,0]} \in \mathcal{B}_\mu$. As $x|_{[0,\omega]}$ does not depend on the particular choice of $\varphi \in \mathcal{B}_\mu$, it follows that $[-1, \infty) \ni t \mapsto x(\omega + t) \in \mathbb{R}$ is also a solution, and $x(\omega + t) = x(t)$ for all $t \geq 0$. Let $q(a, b, c, \mu) \colon \mathbb{R} \to \mathbb{R}$ denote the ω-periodic extension of $x|_{[0,\omega]}$ to $\mathbb{R}$. Then $q(a, b, c, \mu)$ is an ω-periodic solution on $\mathbb{R}$ of Eq. (2.1) with $g = \mathrm{Step}_\mu(a, b, c)$. Define $p(a, b, c, \mu) : \mathbb{R} \to \mathbb{R}$ as

$$p(a, b, c, \mu)(t) = q(a, b, c, \mu)(t) + \frac{1}{\mu}, \quad t \in \mathbb{R}. \tag{2.22}$$

Since $q(a, b, c, \mu)$ is an ω-periodic solution of (2.1) with $g = \mathrm{Step}_{\mu}(a, b, c)$, it follows that $p(a, b, c, \mu)$ is an ω-periodic solution of (1.2) with $f = \mathrm{Step}(a, b, c)$, and $\omega > 0$ is its minimal period. Since the initial segment of $q(a, b, c, \mu)$ is in $\mathcal{B}_{\mu}$, it clearly follows that $p(a, b, c, \mu)|_{[-1,0]} \in \mathcal{A}$.

Suppose now that $\psi \in \mathcal{A}$. Then $\tilde{\psi} \in C$ defined by $\tilde{\psi}(\theta) = \psi(\theta) - 1/\mu$, $\theta \in [-1, 0]$, satisfies $\tilde{\psi}(\theta) \geq 1 - 1/\mu$ for all $\theta \in [-1, 0]$. Let $\tilde{x} : [-1, \infty) \to \mathbb{R}$ denote the solution of (2.1) with $g = \mathrm{Step}_{\mu}(a, b, c)$ and initial segment $\tilde{\psi}$. From Proposition 2.2 we know that there exists a minimal $t_* \geq 0$ such that $\tilde{x}(t_*) = 1 - 1/\mu$. Then the mapping $[-1, 0] \ni \theta \mapsto \tilde{x}(t_* + \theta)$ is in $\mathcal{B}_{\mu}$, and from our construction it is clear that

$$\tilde{x}(t + t_*) = q(a, b, c, \mu)(t) \quad \text{for all } t \geq 0.$$

Consequently,

$$y(t + t_*, \psi, f, \mu) = \tilde{x}(t + t_*) + \frac{1}{\mu} = q(a, b, c, \mu)(t) + \frac{1}{\mu} = p(a, b, c, \mu)(t)$$

for all $t \geq 0$. The proof of Theorem 2.1 is now complete. □

Corollary 2.4 *Under the assumptions of Theorem 2.1, the mean value of the periodic solution $p(a, b, c, \mu)$ is*

$$MV\,[p(a, b, c, \mu)] = \frac{1}{\mu}\left(\frac{a\,(z_2 - z_1) - c\,(\omega - z_3)}{\omega} + 1\right),$$

where z_1, z_2, z_3 are given in (2.7), (2.12), (2.15), respectively, and $\omega = z_4$ is the minimal period, calculated in (2.20).

Proof Let a, b, c, μ be as in Theorem 2.1. It is clear from the definition of $p(a, b, c, \mu)$ in (2.22) that

$$MV\,[p(a, b, c, \mu)] = \frac{1}{\omega}\int_0^{\omega} p(a, b, c, \mu)(u)\mathrm{d}u = \frac{1}{\omega}\int_0^{\omega} q(a, b, c, \mu)(u)\mathrm{d}u + \frac{1}{\mu}, \tag{2.23}$$

where $\omega = \omega\,(a, b, c, \mu)$ is the minimal period. Let

$$I = \int_0^{\omega} q(a, b, c, \mu)\,(t)\,\mathrm{d}t.$$

From (2.3) it follows that if x is a solution of type (i) on an interval $[t_0, t_1]$, then

$$\begin{aligned}\mu\int_{t_0}^{t_1} x(t)\mathrm{d}t &= \mu\int_{t_0}^{t_1}\left(\frac{i}{\mu} + \left(x(t_0) - \frac{i}{\mu}\right)e^{-\mu(t-t_0)}\right)\mathrm{d}t\\ &= i\,(t_1 - t_0) + \left(x(t_0) - \frac{i}{\mu}\right)\left(1 - e^{-\mu(t_1-t_0)}\right)\end{aligned}$$

$$= i\,(t_1 - t_0) + \left(x(t_0) - \frac{i}{\mu}\right) - \left(x(t_1) - \frac{i}{\mu}\right)$$
$$= i\,(t_1 - t_0) + x(t_0) - x(t_1).$$

Applying this formula on the five intervals $[0, 1]$, $[1, z_1 + 1]$, $[z_1 + 1, z_2 + 1]$, $[z_2 + 1, z_3 + 1]$ and $[z_3 + 1, \omega]$, we get the five terms

$$-c + x(0) - x(1),\; x(1) - x(z_1 + 1),\; a(z_2 - z_1) + x(z_1 + 1) - x(z_2 + 1),$$
$$x(z_2 + 1) - x(z_3 + 1),\; -c(\omega - z_3 - 1) + x(z_3 + 1) - x(\omega),$$

respectively. Adding these up and using $x(0) = x(\omega)$, we obtain that

$$\mu I = a(z_2 - z_1) - c(\omega - z_3).$$

This result together with (2.23) completes the proof of the corollary. □

3 Hydra Effect for Step Feedback Functions

The next result states that hydra effect appears in Eq. (1.2) for certain step function nonlinearities.

Theorem 3.1 *One may set $a > 0$, $b \in (0, 1)$, $c \in (0, 1)$ such that for all $\varphi \in \mathcal{A}$, the inequality*

$$MV\left[y\left(\cdot, \varphi, f, \frac{1}{1-b}\right)\right] > MV\left[y(\cdot, \varphi, f, 1)\right] \tag{3.1}$$

holds with $f = \mathrm{Step}(a, b, c)$, and hence hydra effect occurs in (1.2).

The proof of this theorem will follow from the forthcoming proposition.

We select the parameters a, b, c according to Theorem 2.1: $a > 0$, $c \in (0, 1)$ and $b \in (0, b_0(a, c))$. Recall that $p(a, b, c, \mu)(t) = q(a, b, c, \mu)(t) + 1/\mu$ for all $t \in \mathbb{R}$ and $\mu \in [1, 1/(1 - b)]$. Assume that the parameter μ increases, and all the other parameters are kept constant. As the term $1/\mu$ decreases, the mean value of $p(a, b, c, \mu)$ does not necessarily increases even if the mean value of $q(a, b, c, \mu)$ increases. The next proposition shows that we can overcome this difficulty by setting a, b and c properly.

Consider the periodic solutions $p(a, b, c, 1)$ and $p(a, b, c, 1/(1-b))$ of (1.2) with $f = \mathrm{Step}(a, b, c)$ corresponding to the cases $\mu = 1$ and $\mu = 1/(1-b)$, respectively.

Proposition 3.2 *Set $0 < a < 2 - e^{-1}$ arbitrarily. Then $b \in (0, 1)$ and $c > 0$ can be chosen such that $(a, b, c, 1) \in U_4$, $(a, b, c, 1/(1 - b)) \in U_4$ and*

$$MV\left[p\left(a, b, c, \frac{1}{1-b}\right)\right] > MV\left[p\,(a, b, c, 1)\right]. \tag{3.2}$$

Similarly, if $c \in (0, 1)$ is an arbitrarily fixed number, then $a > 0$ and $b > 0$ can be chosen so that the above properties hold.

Proof It is convenient to introduce a new parameter $\tau \in [0, 1]$. Assume that parameter μ is defined as a function of τ and b:

$$[0, 1] \times [0, 1) \ni (\tau, b) \mapsto \mu = \mu(\tau, b) = \frac{1}{1 - \tau b} \in [1, \infty).$$

Note that for all $b \in [0, 1)$, $\tau = 0$ gives $\mu = 1$, and $\tau = 1$ gives $\mu = 1/(1 - b)$.

As before, let ω denote the minimal period of $p(a, b, c, \mu)$ and $q(a, b, c, \mu)$ and let

$$I = \int_0^{\omega} q(a, b, c, \mu)(t)\, \mathrm{d}t.$$

As I and ω are functions of (a, b, c, μ) and $\mu = \mu(\tau, b)$, we can view I and ω as functions of a, b, c and τ, that is, $I = I(a, b, c, \tau)$ and $\omega = \omega(a, b, c, \tau)$. Then the mean value in (2.23) can be written as $MV[p(a, b, c, \mu)] = I/\omega + 1 - \tau b$.

For fixed $a > 0$ and $c \in (0, 1)$, consider the map

$$[0, b_0(a, c)) \ni b \mapsto MV\left[p\left(a, b, c, \frac{1}{1 - \tau b}\right)\right] \in \mathbb{R} \tag{3.3}$$

with $\tau = 0$ and $\tau = 1$. Observe that the value of the function in (3.3) at $\tau = 0,\ b = 0$ is the same as at $\tau = 1,\ b = 0$. Hereinafter we show that, for sufficiently small $c > 0$, the inequality

$$\frac{\partial}{\partial b} MV\left[p\left(a, b, c, \frac{1}{1 - \tau b}\right)\right]\Bigg|_{\tau=1, b=0} > \frac{\partial}{\partial b} MV\left[p\left(a, b, c, \frac{1}{1 - \tau b}\right)\right]\Bigg|_{\tau=0, b=0} \tag{3.4}$$

is valid. We remark that the partial derivative $\partial/\partial b$ at $b = 0$ is understood as a right hand derivative. As we can see from Corollary 2.4, the mean value $MV[p(a, b, c, \mu)]$ is an expression of elementary functions, and thus it is smooth, and the above partial derivatives exist. Then it immediately follows that, for given $a \in (0, 2 - 1/e)$ and $c \in (0, 1)$ such that (3.4) is satisfied, there is $b > 0$ so that (3.2) holds.

Writing $'$ for $\frac{\partial}{\partial b}\,|_{b=0}$, it is sufficient to verify that

$$\left(\frac{I}{\omega} + 1 - \tau b\right)'\Bigg|_{\tau=1} > \left(\frac{I}{\omega} + 1 - \tau b\right)'\Bigg|_{\tau=0},$$

or equivalently, to show that

$$\frac{I'(a, 0, c, 1)\, \omega(a, 0, c, 1) - I(a, 0, c, 1)\, \omega'(a, 0, c, 1)}{\omega^2(a, 0, c, 1)} - 1 \tag{3.5}$$

is greater than

$$\frac{I'(a, 0, c, 0)\, \omega(a, 0, c, 0) - I(a, 0, c, 0)\, \omega'(a, 0, c, 0)}{\omega^2(a, 0, c, 0)}. \tag{3.6}$$

Here we only consider the case when $a \in \left(0, 2 - e^{-1}\right)$ is arbitrary and $c > 0$ is small. It is analogous to handle the case when $c \in (0, 1)$ is chosen arbitrarily, and $a > 0$ is small.

First we compute all terms in (3.5) and (3.6) explicitly, and then complete the proof of the proposition.

Step 1. (introduction of functions v_i.) Let us introduce some auxiliary functions to ease the computations. For (a, b, c, τ), provided that $(a, b, c, \mu) \in U_4$, define $v_i = v_i\,(a, b, c, \tau)$, $i \in \{1, 2, 3, 4\}$, as

$$v_1 = \mu z_1 = \ln \frac{1 - \frac{1-c}{\mu}}{1 - \frac{1-c}{\mu} - b},$$

$$v_2 = \mu\,(z_2 - z_1 - 1) = \ln \frac{\left(1 - \frac{1-c}{\mu} - b\right)\left(e^{-\mu} - \frac{\frac{c}{\mu}}{1-\frac{1-c}{\mu}}\right) - \frac{a}{\mu}}{1 - \frac{a+1}{\mu} - b},$$

$$v_3 = \mu\,(z_3 - z_2) = \ln \frac{1 - \frac{a+1}{\mu} - b}{1 - \frac{a+1}{\mu}},$$

$$v_4 = \mu\,(\omega - z_3 - 1) = \ln \frac{\left(1 - \frac{a+1}{\mu}\right)\left(e^{-\mu} + \frac{\frac{a}{\mu}}{1-\frac{a+1}{\mu}-b}\right) + \frac{c}{\mu}}{1 - \frac{1-c}{\mu}},$$

where $\mu = 1/(1 - \tau b)$. Then

$$\omega = 2 + \frac{1}{\mu}\,(v_1 + v_2 + v_3 + v_4)\,, \tag{3.7}$$

and using Corollary 2.4, it is a straightforward calculation to show that

$$I = \frac{a - c}{\mu} + \frac{a}{\mu^2} v_2 - \frac{c}{\mu^2} v_4. \tag{3.8}$$

First we calculate v_i, $i \in \{1, 2, 3, 4\}$, at $b = 0$:

$$\begin{aligned} v_1\,(a, 0, c, \tau) &= 0, \\ v_2\,(a, 0, c, \tau) &= \ln\left(1 + \frac{c}{a}\left(1 - e^{-1}\right)\right), \\ v_3\,(a, 0, c, \tau) &= 0, \\ v_4\,(a, 0, c, \tau) &= \ln\left(1 + \frac{a}{c}\left(1 - e^{-1}\right)\right). \end{aligned} \tag{3.9}$$

The derivatives v_i' exist and are finite for all $a > 0$, $c \in (0, 1)$ and $\tau \in [0, 1]$:

$$\begin{aligned} v_1'\,(a, 0, c, \tau) &= \frac{1}{c}, \\ v_2'\,(a, 0, c, \tau) &= \frac{(\tau - 1)\,(a + c)\,(e - 1) + a\,(c\tau - e)}{a^2 e + ac\,(e - 1)}, \end{aligned}$$

$$v_3'(a,0,c,\tau) = \frac{1}{a},$$
$$v_4'(a,0,c,\tau) = \frac{-\tau(a+c)(e-1) + c(a\tau - e)}{c^2 e + ac(e-1)}. \tag{3.10}$$

Step 2. (calculation of the terms $I(a,0,c,\tau)$, $I'(a,0,c,\tau)$, $\omega(a,0,c,\tau)$ and $\omega'(a,0,c,\tau)$.) It comes from (3.8) and (3.9) that

$$I(a,0,c,\tau) = a - c + a\ln\left(1 + \frac{c}{a}\left(1 - e^{-1}\right)\right) - c\ln\left(1 + \frac{a}{c}\left(1 - e^{-1}\right)\right).$$

Recall that $1/\mu = 1 - \tau b$. As v_2, v_4 and $1/\mu$ are right differentiable at $b = 0$ for all $a > 0$, $c \in (0,1)$ and $\tau \in [0,1]$, the derivative $I'(a,0,c,\tau)$ also exists for all $a > 0$, $c \in (0,1)$ and $\tau \in [0,1]$. Using the equalities (3.8)–(3.10) and

$$\left(\frac{1}{\mu}\right)' = -\tau \quad \text{and} \quad \left(\frac{1}{\mu^2}\right)' = -2\tau,$$

we deduce that

$$\begin{aligned} I'(a,0,c,\tau) = &- \tau(a-c) \\ &- 2\tau\left\{a\ln\left(1 + \frac{c}{a}\left(1 - e^{-1}\right)\right) - c\ln\left(1 + \frac{a}{c}\left(1 - e^{-1}\right)\right)\right\} \\ &+ \frac{(\tau-1)(a+c)(e-1) + a(c\tau - e)}{ae + c(e-1)} \\ &- \frac{-\tau(a+c)(e-1) + c(a\tau - e)}{ce + a(e-1)}. \end{aligned}$$

It follows from (3.7) and (3.9) that for $b = 0$,

$$\omega(a,0,c,\tau) = 2 + \ln\left(1 + \frac{c}{a}\left(1 - e^{-1}\right)\right) + \ln\left(1 + \frac{a}{c}\left(1 - e^{-1}\right)\right).$$

Similarly, $\omega'(a,0,c,\tau)$ exists, and using (3.7) and (3.10) it can be calculated as

$$\begin{aligned} \omega'(a,0,c,\tau) = &- \tau\left\{\ln\left(1 + \frac{c}{a}\left(1 - e^{-1}\right)\right) + \ln\left(1 + \frac{a}{c}\left(1 - e^{-1}\right)\right)\right\} \\ &+ \frac{(\tau-1)(a+c)(e-1) + a(c\tau - e)}{a^2 e + ac(e-1)} \\ &+ \frac{-\tau(a+c)(e-1) + c(a\tau - e)}{c^2 e + ac(e-1)} \\ &+ \frac{1}{a} + \frac{1}{c}. \end{aligned}$$

Step 3. We are now ready to verify that (3.5) is greater than (3.6). As $\omega(a,0,c,1) = \omega(a,0,c,0)$ and $I(a,0,c,1) = I(a,0,c,0)$, the inequality is equivalent to

$$\begin{aligned}&\left[I'(a,0,c,1) - I'(a,0,c,0)\right]\omega(a,0,c,1) - \omega^2(a,0,c,1)\\ &\quad > I(a,0,c,1)\left[\omega'(a,0,c,1) - \omega'(a,0,c,0)\right].\end{aligned}$$

We examine the behavior of the above inequality as $c \to 0+$ and $a \in \left(0, 2 - e^{-1}\right)$ is fixed.

It is straightforward to show that for any $a > 0$,

$$\lim_{c\to 0+} c \ln\left(1 + \frac{a}{c}\left(1 - e^{-1}\right)\right) = 0.$$

Therefore,

$$\lim_{c\to 0+} I(a,0,c,\tau) = a$$

and

$$\lim_{c\to 0+} I'(a,0,c,\tau) = \tau(1-a) + (\tau - 1)\left(1 - e^{-1}\right) - 1.$$

Also note that

$$\lim_{c\to 0+} \omega(a,0,c,\tau) = \lim_{c\to 0+} \omega'(a,0,c,\tau) = \infty.$$

As

$$\lim_{c\to 0+} \left(I'(a,0,c,1) - I'(a,0,c,0)\right) = 2 - a - e^{-1} > 0,$$

the term $I'(a,0,c,1) - I'(a,0,c,0)$ is positive for $c > 0$ small (note that we use the assumption $a < 2 - e^{-1}$ only here). Hence it suffices to verify the simpler inequality

$$-\omega^2(a,0,c,1) > I(a,0,c,1)\left[\omega'(a,0,c,1) - \omega'(a,0,c,0)\right].$$

Moreover, as $\lim_{c\to 0+} I(a,0,c,\tau) = a$ for all $\tau \in [0,1]$, it is enough to prove that

$$\omega^2(a,0,c,1) < a\left[\omega'(a,0,c,0) - \omega'(a,0,c,1)\right].$$

By substitution we obtain that

$$\left\{2 + \ln\left(1 + \frac{c}{a}\left(1 - e^{-1}\right)\right) + \ln\left(1 + \frac{a}{c}\left(1 - e^{-1}\right)\right)\right\}^2 \tag{3.11}$$

has to be smaller than

$$\frac{-(a+c)(e-1)-ac}{ae+c(e-1)} \tag{3.12}$$

$$+\ a\frac{(a+c)(e-1)-ac}{c^2e+ac(e-1)} \tag{3.13}$$

$$+\ a\ln\left(1+\frac{c}{a}\left(1-e^{-1}\right)\right) \tag{3.14}$$

$$+\ a\ln\left(1+\frac{a}{c}\left(1-e^{-1}\right)\right). \tag{3.15}$$

The truth of this inequality now comes from the following observations.

3.1. Note that the second term in (3.11) and (3.14) converge to 0, as $c \to 0+$:

$$\lim_{c\to 0+} \ln\left(1+\frac{c}{a}\left(1-e^{-1}\right)\right) = 0.$$

The term (3.12) also has a finite limit:

$$\lim_{c\to 0+} \frac{-(a+c)(e-1)-ac}{ae+c(e-1)} = \frac{1-e}{e}.$$

3.2. It is clear that the third term in (3.11) and (3.15) diverge to $+\infty$ as $c \to 0+$. Regarding (3.13), we see that for $c > 0$ small enough,

$$a\frac{(a+c)(e-1)-ac}{c^2e+ac(e-1)} = a\frac{\left(\frac{a}{c}+1\right)(e-1)-a}{ce+a(e-1)}$$
$$> a\frac{\frac{1}{2}\left(\frac{a}{c}+1\right)(e-1)}{2a(e-1)} = \frac{1}{4}\left(\frac{a}{c}+1\right).$$

So (3.13) also diverges to $+\infty$ as $c \to 0+$.

3.3. It follows that we only have to examine how the sum of (3.13) and (3.15) and the square of the third term in (3.11) relate to each other in order to decide whether (3.11) is smaller than the sum in (3.12)–(3.15) for small c. An application of l'Hospital's rule yields that

$$\lim_{c\to 0+} \frac{\frac{a}{c}}{\ln^2\left(1+\frac{a}{c}\left(1-e^{-1}\right)\right)} = \infty,$$

so the ratio of (3.13) and (3.11) also diverges to $+\infty$ as $c \to 0+$.

Therefore (3.11) is smaller than the sum given in (3.12)–(3.15) for all $c > 0$ small enough. This completes the proof. □

Proof of Theorem 3.1 Set the parameters a, b and c as in Proposition 3.2. Let $\psi \in \mathcal{A}$. Let $y^1 = y(\cdot, \psi; f, 1)$ and $y^2 = y(\cdot, \psi; f, 1/(1-b))$ denote the corresponding solutions of (1.2) with $f = \text{Step}(a, b, c)$ for $\mu = 1$ and $\mu = 1/(1-b)$, respectively. By Theorem 2.1, there exist $t_1 \geq 0$ and $t_2 \geq 0$ such that

$$y^1(t + t_1) = p(a, b, c, 1)(t)$$

and

$$y^2(t + t_2) = p\left(a, b, c, \frac{1}{1-b}\right)(t)$$

for all $t \geq 0$. This means that

$$MV[y^1] = MV[p(a, b, c, 1)]$$

and

$$MV[y^2] = MV\left[p\left(a, b, c, \frac{1}{1-b}\right)\right].$$

Therefore the theorem follows from Proposition 3.2. □

4 Hydra Effect for Continuous Feedback Functions

Our next purpose is to discuss the appearance of hydra effect for continuous feedback functions. We define neighbourhoods of the step functions $\text{Step}(a, b, c)$. Fix the parameters a, b and c as in Theorem 3.1. Let $\varepsilon > 0$ be small and $\beta \in (0, b/4)$. We say that $F \in C(\mathbb{R}, \mathbb{R})$ belongs to $\mathcal{N}(\beta, \varepsilon, \text{Step}(a, b, c))$ if

- $|F(y) - (1 + a)| \leq \varepsilon$ for all $y \leq 1 - b - \beta$,
- $1 - \varepsilon \leq F(y) \leq 1 + a + \varepsilon$ provided $1 - b - \beta \leq y \leq 1 - b + \beta$,
- $|F(y) - 1| \leq \varepsilon$ provided $1 - b + \beta \leq y \leq 1 - \beta$,
- $1 - c - \varepsilon \leq F(y) \leq 1 + \varepsilon$ provided $1 - \beta \leq y \leq 1 + \beta$,
- $|F(y) - (1 - c)| \leq \varepsilon$ for all $y \geq 1 + \beta$.

See Fig. 5.

The following theorem states that hydra effect occurs in Eq. (1.2) with certain smooth feedback functions f as well.

Theorem 4.1 *Fix a, b and c as in Theorem 3.1. If $\varepsilon > 0$ and $\beta \in (0, b/4)$ are small enough, then one may choose a Lipschitz continuous F in $\mathcal{N}(\beta, \varepsilon, \text{Step}(a, b, c))$ such that for all $\varphi \in C$ with $\varphi(\theta) \geq 1 + \beta$, $-1 \leq \theta \leq 0$, the inequality*

$$MV\left[y\left(\cdot, \varphi, F, \frac{1}{1-b}\right)\right] > MV\left[y\left(\cdot, \varphi, F, 1\right)\right] \tag{4.1}$$

holds.

The proof of Theorem 4.1 is based on an idea of Walther in [21], see also [9,10,20]. Walther first constructs periodic solutions for the discontinuous equation

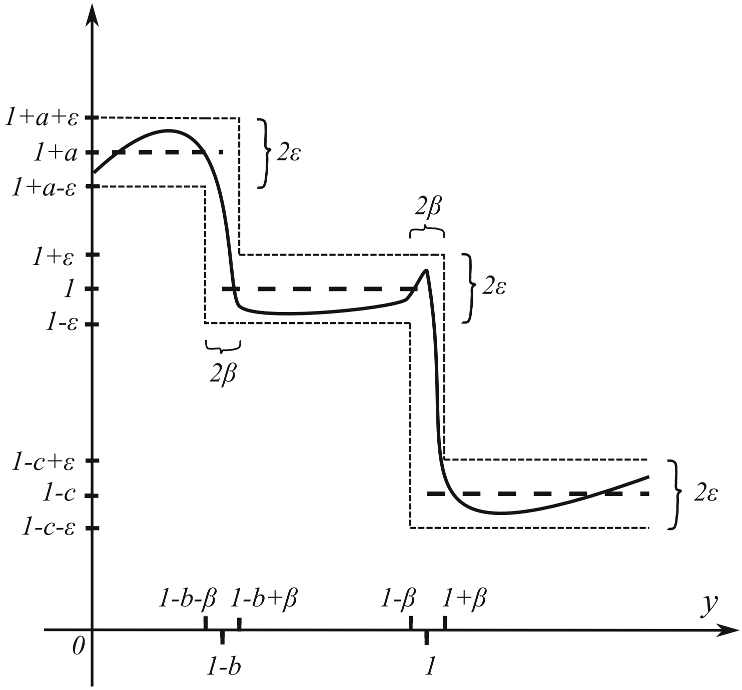


Fig. 5 An element of $\mathcal{N}(\beta, \varepsilon, \mathrm{Step}(a, b, c))$

$$\dot{x}(t) = -\mu x(t) - a\,\mathrm{sign}\,(x(t-1)) \tag{4.2}$$

with $\mu > 0$ and $a > 0$. Then he considers certain Lipschitz continuous functions $F : \mathbb{R} \to \mathbb{R}$ which are close to $-a\,\mathrm{sign}\,(\cdot)$ in the sense that $|F(x) + a\,\mathrm{sign}\,(x)|$ is small for all $|x| \geq \beta$ with some $\beta > 0$ small. He proves the existence of a periodic solution for Eq. (1.2) with such a feedback function F. It is essential that the initial segment of the periodic solution is the fixed point of a Poincaré return map defined on the closed convex set

$$\mathcal{A}_\beta = \{\varphi \in C : \varphi(\theta) \geq \beta \quad \text{for all } \theta \in [-1, 0),\ \varphi(0) = \beta\}.$$

This return map is a strict contraction, and $\mathcal{A}_\beta$ belongs to the stable set of the periodic solution. The paper [21] also guarantees that the periodic solution corresponding to F is arbitrarily close to the periodic solution of (4.2) on bounded intervals of the real line, provided F is sufficiently close to $-a$ sign $(\cdot)$. We cannot guarantee that the two periodic solutions are close to each other on the whole real line as their minimal periods may be different.

Now let $\varepsilon > 0$ and $\beta > 0$ be small, $F \in \mathcal{N}(\beta, \varepsilon, \mathrm{Step}(a, b, c))$ and $\mu \in \{1, 1/(1-b)\}$. The transformed feedback function

$$G_\mu(x) = F(x + 1/\mu) - 1$$

is close to the step function $\mathrm{Step}_\mu(a, b, c)$ introduced in Sect. 2 and plotted in Fig. 3. Observe that here we use the same transformation $x = y - 1/\mu$ as in Sect. 2. The method of Walther [21] can be generalized in a straightforward manner to the equation

$$\dot{x}(t) = -\mu x(t) + G_\mu(x(t-1)).$$

Following [21], we obtain periodic solutions $Q(\mu)$ for both $\mu \in \{1, 1/(1-b)\}$ if β and ε are small enough. In this case the initial segment of $Q(\mu)$ will be the fixed point of a contractive return map defined on $\mathcal{A}_{1-1/\mu+\beta}$ for both $\mu \in \{1, 1/(1-b)\}$. In addition, if β and ε are small enough, then

$$|Q(1)(t) - q(a, b, c, 1)(t)| \quad \text{and} \quad \left|Q\left(\frac{1}{1-b}\right)(t) - q\left(a, b, c, \frac{1}{1-b}\right)(t)\right|$$

are arbitrary small on finite subintervals of the real line for the periodic solutions $q(a, b, c, 1)$ and $q(a, b, c, 1/(1-b))$ defined in Sect. 2. Transforming back, we see that for both $\mu \in \{1, 1/(1-b)\}$, the function $P(\mu)$ defined as

$$P(\mu)(t) = Q(\mu)(t) + \frac{1}{\mu}, \quad t \in \mathbb{R},$$

is a periodic solution of Eq. (1.2) with $f = F$, its initial segment is the fixed point of a contractive return map defined on $\mathcal{A}_{1+\beta}$, and thus $\mathcal{A}_{1+\beta}$ belongs to its stable set. Moreover, on finite subintervals of the real line, $P(1)$ and $P(1/(1-b))$ are arbitrarily close to the periodic solutions $p(a, b, c, 1)$ and $p(a, b, c, 1/(1-b))$, respectively. Therefore, if β and ε are sufficiently small, then Proposition 3.2 guarantees that

$$MV\left[P\left(\frac{1}{1-b}\right)\right] > MV[P(1)].$$

Suppose $\varphi \in C$ with $\varphi(\theta) \geq 1 + \beta$ for all $-1 \leq \theta \leq 0$. Consider the solutions $y(\cdot, \varphi, F, 1/(1-b))$ and $y(\cdot, \varphi, F, 1)$. These solutions will not be, in general, eventually periodic as in the case of the step function nonlinearity. Instead of eventual periodicity it is possible to show that φ is in the stable set of $P(1)$ in the case $\mu = 1$, and φ is in the stable set of $P(1/(1-b))$ in the case $\mu = 1/(1-b)$. In addition, there are $t_1 \geq 0$ and $t_2 \geq 0$ such that

$$\lim_{t\to\infty} |y(t_1 + t, \varphi, F, 1) - P(1)(t)| = 0$$

and

$$\lim_{t\to\infty} |y(t_2 + t, \varphi, F, 1/(1-b)) - P(1/(1-b))(t)| = 0.$$

From these limit properties and from the relation between the mean values of $P(1/(1-b))$ and $P(1)$ it is not difficult to get

$$MV[y(\cdot, \varphi, F, 1/(1-b)] = MV[P(1/(1-b))] > MV[P(1)] \\ = MV[y(\cdot, \varphi; F, 1)],$$

from which Theorem 4.1 follows.

5 Why Does Hydra Effect Appear?

We give a heuristic reasoning why the special form of the feedback function $\mathrm{Step}(a, b, c)$ causes the hydra effect.

Consider the transformed feedback function $\mathrm{Step}_\mu(a, b, c)(x) = \mathrm{Step}(a, b, c)(x + 1/\mu) - 1$ introduced in Sect. 2, see Fig. 3. As the parameter μ increases in the interval $[1, 1/(1-b)]$ and a, b, c are kept constants, the graph of $\mathrm{Step}_\mu(a, b, c)$ is shifted from the left to the right, see Fig. 6. Comparing the two feedback functions $\mathrm{Step}_1(a, b, c)$ and $\mathrm{Step}_{1/(1-b)}(a, b, c)$, the nonincreasing property of our step function implies that

$$\mathrm{Step}_1(a, b, c)(x) \leq \mathrm{Step}_{1/(1-b)}(a, b, c)(x) \quad \text{for all } x \in \mathbb{R},$$

and strict inequality holds on $(-b, 0) \cup (0, b)$. This means that for a term $x(t-1) < 0$ a stronger restoring feedback applies in case $\mu = 1/(1-b)$ than in case $\mu = 1$. If $x(t-1) > 0$ then a weaker restoring feedback applies in case $\mu = 1/(1-b)$ than in case $\mu = 1$. Then if we start a solution from the same initial segment then heuristically it is expected that in average the solution with parameter $\mu = 1/(1-b)$ should be larger than the solution with parameter $\mu = 1$. This may not be enough for hydra effect because, for example for periodic solutions, from Sect. 2 we have the relation

$$\frac{1}{\omega}\int_0^\omega p(a, b, c, \mu)(u)\mathrm{d}u = \frac{1}{\omega}\int_0^\omega q(a, b, c, \mu)(u)\mathrm{d}u + \frac{1}{\mu},$$

between the averages of $p(a, b, c, \mu)$ and $q(a, b, c, \mu)$. The term $1/\mu$ decreases as μ increases. So in order to have hydra effect, the increase in the mean value of $q(a, b, c, \mu)$ must be larger than the decrease of $1/\mu$, as μ increases. In fact, we analytically proved in Sects. 2, 3 that this can be achieved.

Therefore, hydra effect occurs in our equation because the increase of μ deforms the feedback around the equlibrium $1/\mu$ so that the negative feedback on the left side of the equlibrium becomes much stronger, while on the right side it becomes much

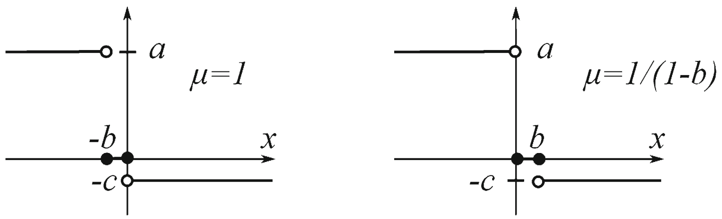


Fig. 6 The plot of $\mathrm{Step}_\mu(a, b, c)$ for $\mu = 1$ and $\mu = 1/(1-b)$

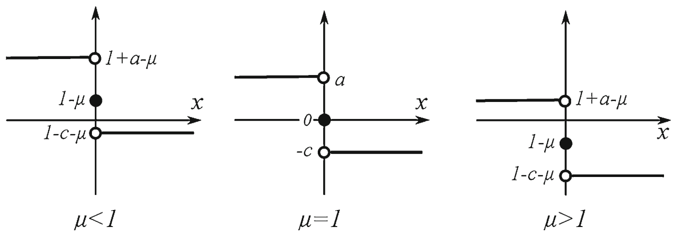


Fig. 7 The plot of h_μ for $\mu < 1$, $\mu = 1$ and $\mu > 1$

weaker, and, in addition, this effect is strong enough to compensate the opposite impact caused by the decrease of the equilibrium $1/\mu$ as a function of μ.

We close the discussion with an example in which hydra effect does not occur. This may also support our heuristic explanation for hydra effect. Consider Eq. (1.2) with $f = \text{Step}(a, 0, c)$ and $a > 0$, $c \in (0, 1)$. Note that 1 is the unique equilibrium if $\mu = 1$, and there is no equilibrium for μ in $(1 - c, 1 + a)\setminus\{1\}$. We can construct periodic solutions $p(a, 0, c, \mu)$ for all μ in a neighborhood of 1. Calculating the mean values of $p(a, 0, c, \mu)$, it turns out that there is no hydra effect.

Although there is no equilibrium for μ in $(1 - c, 1 + a)\setminus\{1\}$, we would have at least one equilibrium in a neighborhood of 1 for continuous feedback functions close to $\text{Step}(a, 0, c)$. So, we use the transformation $y = x + 1$, and consider the equation $\dot{x}(t) = -\mu x(t) + h_\mu(x(t-1))$, where $h_\mu(x) = \text{Step}(a, 0, c)(x+1) - \mu$. The graph of h_μ shifts downwards as μ increases, that is, $h_\mu(x)$ decreases as μ increases for all $x \in \mathbb{R}$, see Fig. 7. We suspect that this is the reason for the decrease of the mean value of the periodic solution.

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