COMBINATORIAL PROPERTIES OF POLY-BERNOULLI RELATIVES

Beáta Bényi
Faculty of Water Sciences, National University of Public Service, Hungary
beata.benyi@gmail.com

Péter Hajnal[1]
Bolyai Institute, University of Szeged, Hungary and
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Hungary
hajnal@math.u-szeged.hu

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Abstract

In this note we augment the poly-Bernoulli family with two new combinatorial objects. We derive formulas for the relatives of the poly-Bernoulli numbers using the appropriate variations of combinatorial interpretations. Our goal is to show connections between the different areas where poly-Bernoulli numbers and their relatives appear and give examples of how the combinatorial methods can be used for deriving formulas between these integer arrays.

1. Introduction

Poly-Bernoulli numbers were introduced by M. Kaneko [24] in 1997 as a generalization of the classical Bernoulli numbers during his investigations of multiple zeta values. A great deal of attention is being paid to this sequence because of its interesting properties, that were analytically proven by several authors. The importance of the notion of the poly-Bernoulli numbers is also underlined by the fact that there are several drastically different combinatorial interpretations [6]. The combinatorics of the family of poly-Bernoulli numbers is highlighted by the bijections that create connections between the alternative definitions. These bijections help us to understand several of the properties of the poly-Bernoulli numbers.

In this paper we consider two number arrays that are relatives of poly-Bernoulli numbers. The importance of this study is that in some combinatorial problems these relatives arise naturally. Kaneko's number-theoretical investigations also led

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to these numbers. We go through the known combinatorial interpretations of poly-Bernoulli numbers [6] and for most of them we show that slight modifications of the original combinatorial definition lead to the descriptions of the two related sequences. We enrich the list of the poly-Bernoulli family with two classes of 01 matrices, permutation tableaux of rectangular shape, and permutations with a given excedance set.

The outline of the paper is as follows. After a short introduction of the poly-Bernoulli numbers, we define the poly-Bernoulli relatives using the well-known interpretation of lonesum matrices. We derive different formulas for these arrays using appropriate combinatorial interpretations. We close our discussion with a conjecture related to the central binomial sum.

1.1. Poly-Bernoulli Numbers

The story of the Bernoulli numbers starts with investigating the sum of the mth powers of the first n positive integers that are polynomials in n. Jacob Bernoulli recognized the scheme in the coefficients of these polynomials. Kaneko generalized the well-known generating function of the Bernoulli numbers and defined the poly-Bernoulli numbers.

Definition 1 ([25]). *Poly–Bernoulli numbers* (denoted by $B_n^{(k)}$, where n is a positive integer and k is an integer) are defined by the following exponential generating function

$$\sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} = \frac{Li_k(1-e^{-x})}{1-e^{-x}}, \tag{1}$$

where

$$Li_k(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^k}.$$

From the combinatorial point of view, we are only interested in the poly-Bernoulli numbers with negative k indices, since, in this case, the numbers form an array of positive integers. From now on, when we refer to poly-Bernoulli numbers, we will always mean poly-Bernoulli numbers with negative indices. For the sake of convenience, we denote in the rest of the paper $B_n^{(-k)}$ as $B_{n,k}$. Table 1 below shows the values of poly-Bernoulli numbers for small indices. An extended array can be found in OEIS [33, A099594].

The symmetry of the array in n and k is immediately noticable. Analytically this property is obvious from the symmetry of the double exponential function (2):

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_{n,k} \frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^{x+y}}{e^x+e^y-e^{x+y}}. \tag{2}$$

n, k	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	4	8	16	32
2	1	4	14	46	146	454
3	1	8	46	230	1066	4718
4	1	16	146	1066	6906	41506
5	1	32	454	4718	41506	329462

Table 1: The poly-Bernoulli numbers $B_{n,k}$

Three formulas of poly-Bernoulli numbers were proven combinatorially in the literature:

1. the combinatorial formula ([8], [6])

$$B_{n,k} = \sum_{m=0}^{\min(n,k)} m! \left\{ {n+1 \atop m+1} \right\} m! \left\{ {k+1 \atop m+1} \right\}, \tag{3}$$

2. the inclusion-exclusion type formula ([8])

$$B_{n,k} = (-1)^n \sum_{m=0}^{n} (-1)^m m! \left\{ {n \atop m} \right\} (m+1)^k, \tag{4}$$

3. and the recurrence relation ([6])

$$B_{n,k+1} = B_{n,k} + \sum_{m=1}^{n} \binom{n}{m} B_{n-(m-1),k}. \tag{5}$$

One of the first (and widely known) combinatorial interpretations of the poly-Bernoulli numbers is the set of lonesum matrices [8]. Lonesum matrices arise in the roots of discrete tomography. In the late 1950's Ryser [35] investigated the problem of the reconstruction of a matrix from given row and column sums. The 01 matrices that are uniquely reconstructible from their row and column sum vectors are called *lonesum* matrices. We denote the set of lonesum matrices of size $n \times k$ as $\mathcal{L}_n^k$. Note that we allow $n = 0$ (and $k = 0$ also), in which cases the empty matrix is counted as lonesum.

Theorem 1 ([8]). *The number of* 01 *lonesum matrices of size* $n \times k$ *is given by the poly-Bernoulli numbers of negative* k *indices.*

$$|\mathcal{L}_n^k| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n+1 \atop m+1} \right\} \left\{ {k+1 \atop m+1} \right\} = B_{n,k}.$$

Proof. (Sketch) Take a lonesum matrix M of size $n \times k$. Add a new column and a new row with all 0 entries and obtain lonesum matrix $\widehat{M}$ of size $(n+1) \times (k+1)$. We know that $\widehat{M}$ contains at least one all-0 row and at least one all 0 column (this information was not known for M). Partition the rows and the columns according to the sum of their entries. In the case of lonesum matrices 'having the same row/column sum' and 'being equal' is the same relation. Easy to see that the number of row classes will be the same as the number of equivalence classes of columns. We denote this common value by $m+1$. The 'plus one' stands for the class of extra row/column, the class of all 0 rows and all-0 columns. The row sums order the (m many) classes of not all 0 rows. Similarly, the column sums order the classes of not all 0 columns. Decoding M is easy when it is based on two partitions and two orders. □

This important theorem started the combinatorial investigations of poly-Bernoulli numbers. It turned out that there are several alternative combinatorial ways to describe the poly-Bernoulli numbers. Some of them were investigated before Kaneko's pioneering work. Next we define two related 2-dimensional sequences combinatorially.

1.2. PB-Relatives

We consider lonesum matrices with further restrictions on the occurrence of all 0 columns resp. all 0 rows. More precisely, let $\mathcal{L}_n^k(c|)$ denote the set of lonesum matrices with the property that each column contains at least one 1 entry and let $\mathcal{L}_n^k(c|r|)$ denote the set of lonesum matrices with the property that each column and each row contains at least one 1 entry.

Definition 2. We denote the number of lonesum matrices of size $n \times k$ without all 0 columns ($|\mathcal{L}_n^k(c|)|$) by $C_{n,k}$. We denote the number of lonesum matrices of size $n \times k$ without all 0 columns and all 0 rows ($|\mathcal{L}_n^k(c|r|)|$) by $D_{n,k}$.

First we give the combinatorial formulas for these two new number sequences.

Theorem 2. *We have*

(i) for $n \geq 1$ and $k \geq 0$

$$C_{n,k} = |\mathcal{L}_n^k(c|)| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n+1 \atop m+1} \right\} \left\{ {k \atop m} \right\},$$

(ii) for $n \geq 1$ and $k \geq 1$

$$D_{n,k} = |\mathcal{L}_n^k(c|r|)| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n \atop m} \right\} \left\{ {k \atop m} \right\}.$$

Proof. (Sketch) (i): Since we know that there is no all 0 column, we do not need the extra column. The extra row ensures that the extended matrix has the class of all 0 rows. As m denotes the number of classes of (k many non-0) columns, $m+1$ will be the number of classes of the $n+1$ rows. The rest is a straightforward repeat of the original argument.

(ii) It follows immediately by the same logic. □

Let us see the first few values of our new numbers:

n,k	0	1	2	3	4
1	1	1	1	1	1
2	1	3	7	15	31
3	1	7	31	115	391
4	1	15	115	675	3451
5	1	31	391	3451	25231

n,k	1	2	3	4	5
1	1	1	1	1	1
2	1	5	13	29	61
3	1	13	73	301	1081
4	1	29	301	2069	11581
5	1	61	1081	11581	95401

Table 2: Poly–Bernoulli relatives: $C_{n,k}$ and $D_{n,k}$

Though these numbers seem to be just minor modifications of poly–Bernoulli numbers, it turned out that they already appeared in earlier papers.

From the combinatorial definitions it is obvious that the sequences $B_{n,k}$ and $D_{n,k}$ are symmetric in n and k.

Since the symmetry of the C-relatives, $C_{n,k}$, ($C_{n,k} = C_{k+1,n-1}$) does not follow from the definition, we present proof for it in a latter section of this paper.

The combinatorial definitions make it clear that the sequence $C_{n,k}$ is the binomial transform of $D_{n,k}$, and the sequence $B_{n,k}$ is the binomial transform of $C_{n,k}$.

Observation 1. *The following identities hold:*

(i)

$$B_{n,k} = 1 + \sum_{i=1}^{k} \binom{k}{i} C_{n,i} = \sum_{i=0}^{k} \binom{k}{i} C_{n,i}, \qquad (k \geq 0, n \geq 1),$$

(ii)

$$C_{n,k} = \sum_{i=1}^{n} \binom{n}{i} D_{i,k}, \qquad (k \geq 1, n \geq 1),$$

(iii)

$$B_{n,k} = 1 + \sum_{i=1}^{n} \sum_{j=1}^{k} \binom{n}{i} \binom{k}{j} D_{i,j}, \qquad (k \geq 1, n \geq 1).$$

Proof. (i): In order to describe an arbitrary non-0 lonesum matrix, we need to identify all its columns with at least one 1 (their number is denoted by $i(>0)$) as well as their entries in these i columns (that is describing a lonesum matrix of size $n \times i$ that contains at least one 1 in each column). This simple fact proves the formula (i).

We obtain (ii) by following the same argument on the rows. Equation (iii) summarizes (i) and (ii). □

We discuss other connections, recurrence relations, and combinatorial properties of the pB–relatives $C_{n,k}$ and $D_{n,k}$ using appropriate combinatorial interpretations; however, first we summarize the analytical properties of these numbers (obtained by other authors).

1.3. Analytical Results in the Literature

Arakawa and Kaneko [4] introduced a function that is referred to in the literature as the Arakawa-Kaneko function:

$$\xi_k(s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} Li_k(1 - e^{-t}) dt.$$

The values of this function at non-positive integers are given by

$$\xi_k(-n) = (-1)^n C_n^{(k)},$$

where the generating function of the numbers $\{C_n^{(k)}\}$ (for arbitrary integers k) is given by

$$\sum_{n=0}^{\infty} C_n^{(k)} \frac{x^n}{n!} = \frac{Li_k(1 - e^{-x})}{e^x - 1}.$$

The exponential functions of two number sequences differ only by an e^x (resp. e^y) factor, when one sequence is the binomial transform of the other. From this observation we can conclude the relation between poly-Bernoulli numbers and $\{C_n^{(-k)}\}$ and it is immediately clear that $C_n^{(-k)} = C_{n,k}$. Furthermore, we obtain the generating function of $D_{n,k}$ numbers.

Theorem 3. *For all positive integers n and k*

(i)

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} C_{n,k} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^x}{e^x + e^y - e^{x+y}},$$

(ii)

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} D_{n,k} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{1}{e^x + e^y - e^{x+y}}.$$

Kaneko also showed a simple arithmetic connection between the two sequences [26]:

$$B_{n,k} = C_{n,k} + C_{n+1,k-1}.$$

In our investigations we show this equality combinatorially using the variations of the so called Callan permutations. Moreover, we prove a similar identity involving the numbers $D_{n,k}$ and $C_{n,k}$.

2. The 01 Matrices With Excluded Submatrices

The study of matrices that are characterized by excluded submatrices is an active research area with many important results and applications [31]. Given two matrices A and B, we say that A *avoids* B whenever A does not contain B as a submatrix. (Given a matrix M, a submatrix of M is a matrix that can be obtained from M by deletion of rows and columns.)

Generally we can set the following problem: Let $S = \{M_1, \ldots, M_r\}$ be a set of 01 matrices. Let $\mathcal{M}_n^k(S)$ denote the set of $n \times k$ 01 matrices that do not contain any matrix of the set S, let $\mathcal{M}_n^k(S; c|)$ denote these matrices with the extra condition of containing in any column at least one 1, and let $\mathcal{M}_n^k(S; r|c|)$ denote those with the same extra condition on rows too.

Lonesum matrices can be characterized also with the terminology of forbidden submatrices [35]. Lonesum matrices are matrices that avoid the following set of submatrices:

$$L = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Thus $\mathcal{L}_n^k = \mathcal{M}_n^k(L)$, $\mathcal{L}_n^k(c|) = \mathcal{M}_n^k(L; c|)$ and $\mathcal{L}_n^k(r|c|) = \mathcal{M}_n^k(L; r|c|)$.

Interestingly, beyond the set L there are other sets S with $|\mathcal{M}_n^k(S)| = B_{n,k}$.

2.1. The Γ-free Matrices and the Recurrence Relations

In [6] the authors investigated the so called Γ-free matrices, matrices with the forbidden set:

$$\Gamma = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\},$$

and showed bijectively that the number of $n \times k$ Γ-free matrices (their set is denoted by $\mathcal{G}_n^k$) are the $B_{n,k}$ poly-Bernoulli numbers. Clearly, the forbiddance of all 0 rows (resp. columns) has the same effect in this case as in the case of the lonesum matrices.

Theorem 4. *We have*

(i)

$$|\mathcal{G}_n^k(c|)| = C_{n,k},$$

(ii)

$$|\mathcal{G}_n^k(r|c|)| = D_{n,k}.$$

The structure of these matrices gives a transparent explanation of the recursive formula of poly-Bernoulli numbers that was first proven by Kaneko [4]. In the same spirit we can establish the recurrence relations concerning the poly-Bernoulli relatives.

Theorem 5. *The following recursive relations hold:*

(i)

$$C_{n,k+1} = \sum_{m=1}^{n} \binom{n}{m} C_{n-m+1,k}, \qquad n \geq 1, \quad k \geq 0,$$

(ii)

$$D_{n,k+1} = \sum_{m=1}^{n} \binom{n}{m} (D_{n-m,k} + D_{n-m+1,k}), \qquad n \geq 1, \quad k \geq 1.$$

Proof. (i): $C_{n,k+1}$ counts the Γ-free matrices of size $n \times (k+1)$ without all-0 column. Each row of a Γ-free matrix falls into exactly one of the following three types

A. starts with 0

B. starts with 1 followed only by 0s

C. starts with 1 and contains at least one more 1.

Let m denote the number of rows that start with 1. Since all columns contain at least one 1, we have $m \geq 1$. We can choose these m rows in $\binom{n}{m}$ different ways. The first $m-1$ rows have to be of type B; otherwise a Γ would appear. The remaining $(n-m+1) \times k$ elements can be filled with an arbitrary Γ-free matrix that contains in any column at least one 1.

(ii): If we argue the same way as before, we obtain

$$\sum_{m=1}^{n} \binom{n}{m} D_{n-m+1,k},$$

but we do not count matrices that contain only type A and type B rows (and do not have type C rows). In this case the remaining $(n-m+1) \times k$ elements contain at least one all 0 row (the remainder of a type B row). Hence, these matrices are not counted in the above formula.

To correct the enumeration (to count the missing matrices) we must add the term

$$\sum_{m=1}^{n} \binom{n}{m} D_{n-m,k},$$

and obtain the recurrence relation (ii). □

2.2. Permutation Tableaux of Size $n \times k$

Permutation tableaux were introduced by Postnikov [34] during his investigations of totally Grassmannian cells. They received a lot of attention after Viennot [41] showed their one-to-one correspondence to permutations, alternative tableaux, and the strong connection to the PASEP model in statistical mechanics. Bijections were used to enumerate permutations according to certain statistics [16], [11].

Permutation tableaux are usually defined as 01 fillings of Ferrers diagrams with the next two conditions:

- each column contains at least one 1.
- each cell with at least one 1 above in the same column and to its left in the same row must contain the entry 1.

In the special case when the Ferrers diagram is an $n \times k$ array, the definition is equivalent to the definition of the set $\mathcal{M}_n^k(P; c|)$ with

$$P = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Theorem 6. *We have*

(i)

$$|\mathcal{M}_n^k(P)| = B_{n,k},$$

(ii)

$$|\mathcal{M}_n^k(P; c|)| = C_{n,k},$$

(iii)

$$|\mathcal{M}_n^k(P; r|c|)| = D_{n,k}.$$

Proof. The first statement (i) is contained in [28] without the recognition of the relation to the poly-Bernoulli numbers. In [42] in Lemma 4.3.5 the author also proves the formula and as a corollary he recognized that the number of $n \times k$ patterns of permutation diagrams is the poly-Bernoulli numbers $B_{n,k}$. For details see [42].

(ii) and (iii) are proven by the obvious binomial correspondences between $|\mathcal{M}_n^k(P)|$, $|\mathcal{M}_n^k(P; c|)|$, and $|\mathcal{M}_n^k(P; r|c|)|$. □

This theorem evolved from a certain bijection between permutations and permutation tableaux that we cite in a latter section.

We see that in the case of permutation tableaux the important variant is the C-relative, the one that corresponds to the restriction of the columns. This is one of the reasons why we think that the introduction and investigation of the variants of poly-Bernoulli numbers is useful.

2.3. A Further Excluded Submatrix Set

Brewbaker made extensive computations considering 01 matrices with excluded patterns [9]. These computations suggest the following theorem.

Theorem 7. *Let Q be the set*

$$Q = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

Then we have

(i)

$$|\mathcal{M}_n^k(Q)| = B_{n,k},$$

(ii)

$$|\mathcal{M}_n^k(Q; c|)| = C_{n,k},$$

(iii)

$$|\mathcal{M}_n^k(Q; r|c|)| = D_{n,k}.$$

Proof. (i): Let M be a matrix in $\mathcal{M}_n^k(Q)$ and $Q_{n,k} = |\mathcal{M}_n^k(Q)|$. Let $j_1 > j_2 > \cdots > j_m$ be the indices of the rows of the 1 entries in the first column ($m \geq 0$). When $m = 0$ or $m = 1$, the first column does not restrict the remaining $n \times (k-1)$ entries; hence, it can be filled with an arbitrary matrix in $\mathcal{M}_n^{k-1}(Q)$. When $m \geq 2$ the rows $j_1, j_2, \ldots j_m$ are identical; otherwise one of the submatrices in Q would appear. It is enough to describe a $(n-m+1) \times (k-1)$ Q-free matrix in order to define M. We have:

$$Q_{n,k} = (n+1)Q_{n,k-1} + \sum_{m=2}^{n} \binom{n}{m} Q_{n-m+1,k-1}.$$

Hence, the $Q_{n,k}$ numbers and the poly-Bernoulli numbers satisfy the same recurrence relation. Induction proves (i).

(ii), (iii): The above proof of recurrence relation easily extends to show the corresponding recurrence relations for $|\mathcal{M}_n^k(Q; c|)|$ and $|\mathcal{M}_n^k(Q; r|c|)|$. □

3. Permutations

In this section we consider classes of permutations that are enumerated by the poly-Bernoulli numbers as well as their relatives. As usual let $\{1, \ldots, n\} = [n]$ and S_n denote the set of permutations of $[n]$.

3.1. Vesztergombi Permutations

The permutations we consider in this section are permutations that are restricted by constraints on the distance between the elements and their images. The enumeration of such permutation classes is a special case of a more general problem setting that were investigated by many authors. Given n subsets $A_1, A_2, \ldots A_n$ of $[n]$, determine the number of permutations π such that $\pi(i) \in A_i$ for all $i \in [n]$. The problem can be formulated as the enumeration of 1–factors of a bipartite graph, as the determination of the permanent of a 01 matrix, or as the number of rook-placements of a given board. In general, these formulations do not make the problem easier.

We want to use the results of Lovász and Vesztergombi ([40],[32], [30]) for derivation of the inclusion-exclusion type formulas for $C_{n,k}$ and $D_{n,k}$. We recall definitions and main ideas for the sake of clearity. Detailed combinatorial proofs and analytical derivations can be found in the cited articles.

Let $f(r, n, k)$ denote the number of permutations $\pi \in S_{n+k}$ satisfying

$$-(k + r) < \pi(i) - i < n + r.$$

Theorem 8 ([40]). *For all non-negative integers n, k and r the following holds:*

$$f(r, n, k) = \sum_{m=0}^{n} (-1)^{n+m} (m + r)!(m + r)^k \left\{ {n+1 \atop m+1} \right\}.$$

The original proof is analytic and depends on the solution of certain differential equations for a generating function based on the $f(r, n, k)$ numbers. The differential equations capture the recurrence relations that follow from the expanding rules of the corresponding permanent.

Launois [29] realized the connection of this formula to the poly-Bernoulli numbers, namely that $f(2, n - 1, k - 1) = B_{n,k}$.

We call a permutation π of $[n + k]$ *Vesztergombi permutation* if

$$-k \leq \pi(i) - i \leq n$$

for all $i \in [n + k]$.

Theorem 9 ([29]). *Let $\mathcal{V}_n^k$ denote the set of Vesztergombi permutations. Then*

$$|\mathcal{V}_n^k| = B_{n,k}.$$

Beyond the analytical derivation of the formula one can find combinatorial proofs of the theorem in the literature. In [27] the authors define an explicit bijection between Vesztergombi permutations and lonesum matrices. In [30] we find a combinatorial proof for a more general case that includes the theorem. For the sake of completeness we present here the direct combinatorial proof from [7].

Proof. (Theorem 9.) The $|\mathcal{V}_n^k|$ is the permanent of the $(n+k)\times(n+k)$ matrix $A=(a_{ij})$, where

$$a_{ij}=\begin{cases} 1 & \text{if} \quad -k\le i-j\le n, \quad i=1,\dots,n+k \\ 0 & \text{otherwise.}\end{cases}$$

The matrix A is built up of 4 blocks, two all 1 matrices $(J_{n,k}, J_{k,n})$, an upper (T_n) and a lower (T^k) triangular matrix. Precisely, let A be the matrix:

$$A=\begin{bmatrix} J_{n,k} & T_n \\ T^k & J_{k,n}\end{bmatrix},$$

where $J_{n,k}\in\{0,1\}^{n\times k}$: $J_{n,k}(i,j)=1$ for all i,j, $J_{k,n}\in\{0,1\}^{k\times n}$: $J_{k,n}(i,j)=1$ for all i,j, $T_n\in\{0,1\}^{n\times n}$: $T_n(i,j)=1$ if and only if $i\ge j$, and $T^k\in\{0,1\}^{k\times k}$: $T^k_{ij}=1$ if and only if $i\le j$.

For a term in the expansion of the permanent we have to select exactly one 1 from each row and each column. The number of ways of selecting 1s from the triangular matrices is given by the Stirling number of the second kind. (See proof for instance in [30].) So if a term contains m 1's from the upper left block $J_{n,k}$ (in $m!$ different ways), then it contains $n-m$ 1's from T_n (in $\left\{{n+1 \atop m+1}\right\}$ different ways); m 1's from the lower right block $J_{k,n}$ (in $m!$ different ways) and finally $k-m$ 1's from T^k (in $\left\{{k+1 \atop m+1}\right\}$ different ways). The total number of terms in the expansion of the permanent of A is

$$\sum_{m=1}^{\min(n,k)} m!\left\{{n+1 \atop m+1}\right\} m!\left\{{k+1 \atop m+1}\right\}.$$

This proves the theorem. □

Suitable modifications of the definition of Vesztergombi permutations lead to the pB–relatives. Let $\mathcal{V}_n^{k*}$ denote the set of permutations π of $[n+k]$ such that

$$-k\le \pi(i)-i<n \quad \text{for all} \quad i\in[n+k],$$

and let $\mathcal{V}_n^{k**}$ denote the set of permutations π of $[n+k]$ such that

$$-k<\pi(i)-i<n \quad \text{for all} \quad i\in[n+k].$$

Theorem 10 ([40]). *The number of the modified Vesztergombi permutations $\mathcal{V}_n^{k*}$ and $\mathcal{V}_n^{k**}$ are given by the poly–Bernoulli relatives.*

(i)
$$|\mathcal{V}_n^{k*}| = C_{n,k},$$

(ii)
$$|\mathcal{V}_n^{k**}| = D_{n,k}.$$

Proof. In these cases the blocks T_n, T^k are slightly changed. The modifications are straightforward; hence, the details are omitted. □

Corollary 1. *The inclusion-exclusion type formulas of the poly–Bernoulli relatives are as follows*

(i)
$$C_{n,k} = \sum_{m=0}^{n} (-1)^{n+m} m!(m+1)^k \left\{ {n+1 \atop m+1} \right\},$$

(ii)
$$D_{n,k} = \sum_{m=0}^{n} (-1)^{n+m} m! m^k \left\{ {n+1 \atop m+1} \right\}.$$

Proof. Clearly $|\mathcal{V}_n^{k*}| = f(1, n, k-1)$ and $|\mathcal{V}_n^{k**}| = f(0, n, k)$. □

In [32] Theorem 1 describes the asymptotic behavior of $D_{n,n}$.

Theorem 11 ([32]). *The asymptotic of $D_{n,n}$ for all $n \geq 1$ is given by the following formula*
$$D_{n,n} \sim \sqrt{\frac{1}{2\pi(1-\ln 2)}} (n!)^2 \frac{1}{(\ln 2)^{2n}}.$$

3.2. Permutations With Excedance Set $[k]$

Permutations that have special restrictions on their excedance set are enumerated by the poly-Bernoulli numbers as well as their relatives. We note that the connection of this class of permutations to poly-Bernoulli numbers is not mentioned directly in the literature.

We call an index i an *excedance* (resp. *weak excedance*) of the permutation π when $\pi(i) > i$ (resp. $\pi(i) \geq i$). According to this, we define the set of excedances (resp. the set of weak excedances) of a permutation π as $E(\pi) := \{i | \pi(i) > i\}$ and

$WE(\pi) := \{i|\pi(i) \geq i\}$. Further, let us define the following sets of permutations of $[n+k]$ with conditions on their excedance sets:

$$\begin{aligned}
\mathcal{E}_n^k &:= \{\pi|\pi \in S_{n+k} \text{ and } WE(\pi) \supset [k] \text{ and } \pi(i) = i \text{ if } i \in WE(\pi) \text{ and } k < i \leq n+k\},\\
\mathcal{E}_n^{k*} &:= \{\pi|\pi \in S_{n+k} \text{ and } E(\pi) = [k]\},\\
\mathcal{E}_n^{k**} &:= \{\pi|\pi \in S_{n+k} \text{ and } E(\pi) = [k] \text{ and } \pi(i) \neq i \text{ for all } 1 \leq i \leq n+k\}.
\end{aligned}$$

The main result in this line of research is summarized in the next theorem.

Theorem 12. *The following three statements hold:*

(i)
$$|\mathcal{E}_n^k| = B_{n,k},$$

(ii)
$$|\mathcal{E}_n^{k*}| = C_{n,k},$$

(iii)
$$|\mathcal{E}_n^{k**}| = D_{n,k}.$$

Proof. There are trivial bijections between these permutations and the three variants of Vesztergombi permutations. We obtain the underlying matrices of the permutation classes $\mathcal{E}_n^k$, $\mathcal{E}_n^{k*}$, and $\mathcal{E}_n^{k**}$ by shifting the building blocks of the underlying matrix A of the appropriate variant of the Vesztergombi permutation. We just sketch the necessary ideas for (i).

In this case we need to compute the permanent of the following matrix E:

$$E = \begin{bmatrix} T_k & J_{k,n} \\ J_{n,k} & T^n \end{bmatrix},$$

where $J_{n,k} \in \{0,1\}^{n\times k}$ and $J_{k,n} \in \{0,1\}^{k\times n}$ are as above (the all 1 matrices), $T_k \in \{0,1\}^{k\times k}$: $T_k(i,j) = 1$ if and only if $i \geq j$, and $T^n \in \{0,1\}^{n\times n}$: $T^n(i,j) = 1$ if and only if $i \leq j$.

The terms in the expansion of per$E$ can be bijectively identified with the terms in the corresponding expansion of the permanent A. □

Next we connect $\mathcal{E}_n^k$ in another way to the poly-Bernoulli family; hence, we give an alternative proof of (i).

As previously mentioned, permutation tableaux are well-studied objects and several bijections are known between permutations and permutation tableaux. We describe here a bijection that is a bijection between the sets $\mathcal{M}_n^k(P)$ and $\mathcal{E}_n^k$ when applying it to the subset of permutation tableaux with rectangular Ferrers shapes.

Theorem 13. *The following statement holds:*

$$|\mathcal{M}_n^k(P)| = |\mathcal{E}_n^k|.$$

We modify the bijection given in [11] in order to have the following properties: excedances of the permutation correspond to column labels and fixed points of the permutation to labels of empty rows. These modifications do not change the bijection essentially.

Proof. (Sketch) Consider an $n \times k$ 01 matrix that avoids the submatrices in the set P and contains at least one 1 in any column. We assign a permutation to this matrix the following way:

Label the positions of the rows from left to right by $[k]$, the positions of the columns from bottom to top by $[n]$. We define the zig-zag path by bouncing right or down every time we hit a 1. For i we find $\pi(i)$ by starting at the top of the column i (the left of the row i) following the zig-zag path until the boundary where we hit the row or column labeled by j. We set $\pi(i) = j$.

The defined map gives a bijection between the two sets in the theorem. □

In [2] the authors investigated the number of the extremal excedance set statistic, i.e., the asymptotic of $C_{n,n}$.

Theorem 14 ([2]). *The asymptotic of $C_{n,n}$ for all $n \geq 1$ is given by the following formula:*

$$C_{n,n} \sim \left(\frac{1}{2\log 2\sqrt{(1-\log 2)}} + o(1) \right) \left(\frac{1}{2\log 2} \right)^{2n} (2n)!.$$

3.3. Callan Permutations

Callan gave an alternative description of poly-Bernoulli numbers in a note in OEIS [33]. As Callan permutations play an important role in proving combinatorial properties of pB-relatives, his description is repeated in Theorem 15.

Definition 3. *Callan permutations* are the permutations of $[n+k]$ in which each substring whose support belongs to $N = \{1, 2, \ldots, n\}$ or $K = \{n+1, n+2, \ldots, n+k\}$ is increasing.

We call the elements in N *left-value elements* and those of K *right-value elements*. For the sake of convenience we rewrite $K \equiv \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}\}$ ($N = \{1, 2, \ldots, n\}$). Actually, we need just the distinction between the elements of the sets N and K and an order in N and K.

Let $\mathcal{C}_n^k$ denote the set of Callan permutations.

Theorem 15. *For all $n \geq 1$ and $k \geq 0$ we have*

$$|\mathcal{C}_n^k| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n+1 \atop m+1} \right\} \left\{ {k+1 \atop m+1} \right\} = B_{n,k}.$$

Proof. (Sketch) Let $\pi \in \mathcal{C}_n^k$. Let $\widetilde{\pi} = 0\pi(\mathbf{k}+\mathbf{1})$, where 0 is a new left value and $\mathbf{k}+\mathbf{1}$ is a new right value. Divide $\widetilde{\pi}$ into maximal blocks of consecutive elements in such a way that each block is a subset of $\{0\} \cup N$ (left blocks) or a subset of $K \cup \{\mathbf{k}+\mathbf{1}\}$ (right blocks). The partition starts with a left block (the block of 0) and ends with a right block (the block of $\mathbf{k}+\mathbf{1}$). So the left and right blocks alternate, and their number is the same, say $m+1$. Describing a Callan permutation is equivalent to specifying m, a partition $\Pi_{\widehat{N}}$ of $\widehat{N} = \{0\} \cup N$ into $m+1$ classes (one class is the class of 0, the other m classes are called ordinary classes), a partition $\Pi_{\widehat{K}}$ of $\widehat{K} = K \dot{\cup} \{\mathbf{k}+\mathbf{1}\}$ into $m+1$ classes (m many of them not containing $\mathbf{k}+\mathbf{1}$, these are the ordinary classes), and two orderings of the ordinary classes. This proves the claim of Callan. □

The role of 0 and $\mathbf{k}+\mathbf{1}$ were important. With the help of them we had the information of how the left and right blocks follow each other.

Let $\mathcal{C}_n^k(*,l)$ be the set of Callan permutations of $N \dot{\cup} K$ that end with a left-value element (and hence with a left block). The star is to remind the reader that there is no condition on the leading block of our permutation. Similarly, let $\mathcal{C}_n^k(l,*)$ be the set of Callan permutations of $N \dot{\cup} K$ that start with a left-value element. Let $\mathcal{C}_n^k(l,r)$ be the set of Callan permutations of $N \dot{\cup} K$ that start with a left-value element and end with a right element. We define the sets analogously: $\mathcal{C}_n^k(r,*)$, $\mathcal{C}_n^k(*,r)$, $\mathcal{C}_n^k(r,l)$, $\mathcal{C}_n^k(l,l)$, and $\mathcal{C}_n^k(r,r)$.

If we take a Callan permutation and reverse the order of its blocks (leaving the order within each block) we obtain a Callan permutation too. This simple observation proves the following equalities:

$$|\mathcal{C}_n^k(*,l)| = |\mathcal{C}_n^k(l,*)|,$$

$$|\mathcal{C}_n^k(r,l)| = |\mathcal{C}_n^k(l,r)|.$$

Now we state our next theorem that gives a new interpretation of pB-relatives with the help of Callan permutations.

Theorem 16. *We have*

(i) for all $n \geq 1$ and $k \geq 0$

$$C_{n,k} = |\mathcal{C}_n^k(*,l)| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n+1 \atop m+1} \right\} \left\{ {k \atop m} \right\},$$

(ii) for all $n \geq 1$ and $k \geq 1$

$$D_{n,k} = |\mathcal{C}_n^k(l,r)| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n \atop m} \right\} \left\{ {k \atop m} \right\}.$$

Proof. (i): Take a $\pi \in \mathcal{C}_n^k(*,l)$ and extend it with a starting 0 (an extra left value): $\widehat{\pi} = 0\pi$. One extra element is enough to control the structure of blocks, because $\widehat{\pi}$ starts and ends with a left block. Let $m+1$ be the number of left blocks, m is the number of right blocks (and the number of ordinary left blocks, i.e., blocks not containing 0). The rest of the proof is a straightforward modification of the previous one.

(ii): Without any extra element we control the starting and ending block. m denotes the common number of left and right blocks. □

The next lemma is implicit in [6]. Since it is central for us, we present it here.

Lemma 2. *There is a bijection*

$$\varphi : \mathcal{C}_n^k(*,l) \to \mathcal{C}_{n-1}^{k+1}(*,r).$$

Proof. Take any $\pi \in \mathcal{C}_n^k(*,l)$. Find n (the largest left value) in it. It is the last element of one of the left blocks (possibly the very last element of π).

Assuming that n is not the last element of π, then it is followed by a right block R and by at least one left block. Exchange n to $\mathbf{k+1}$ and move R to the end of π. The permutation that we obtain this way will be $\varphi(\pi)$.

If n is the last element of π, then exchange it to $\mathbf{k+1}$.

In both cases the described image is obviously in $C_{n-1}^{k+1}(*,r)$. In order to see that φ is a bijection, we need to construct its inverse. This can be done easily based on $\mathbf{k+1}$. □

The lemma gives us a $\psi : \mathcal{C}_n^k(*,r) \to \mathcal{C}_{n+1}^{k-1}(*,l)$ bijection too.

In [6] this lemma was used to prove that $\sum_{k,\ell:k+\ell=n}(-1)^k B_{k,\ell} = 0$. We use the lemma for different purposes. First we prove the symmetry of the $C_{n,k}$ numbers.

Corollary 2. *The array $C_{n,k}$ has the symmetry property:*

$$C_{n,k} = C_{k+1,n-1}.$$

Proof. Change the role of left and right values. The two orderings remain; hence, we obtain a Callan permutation (the blocks remain the same). This leads to a bijection between $\mathcal{C}_n^k(*,l)$ and $\mathcal{C}_k^n(*,r)$. Using the previous lemma we obtain that

$$C_{n,k} = |\mathcal{C}_n^k(*,l)| = |\mathcal{C}_k^n(*,r)| = |\mathcal{C}_{k+1}^{n-1}(*,l)| = C_{k+1,n-1}.$$

□

The next application of our lemma will be a simple connection between poly-Bernoulli numbers and its C-relative. It was proven in [26] with analytical methods. Here we present a combinatorial proof.

Theorem 17 ([26]). *The following arithmetic relation between poly-Bernoulli numbers and poly-Bernoulli C-relatives holds*

$$B_{n,k} = C_{n,k} + C_{k,n} = C_{n,k} + C_{n+1,k-1}.$$

Proof. We know that $B_{n,k} = |\mathcal{C}_n^k|$ and $\mathcal{C}_n^k = \mathcal{C}_n^k(*,l)\dot{\cup}\mathcal{C}_n^k(*,r)$. We have a bijection between $\mathcal{C}_n^k(*,r)$ and $\mathcal{C}_{n+1}^{k-1}(*,l)$. Hence,

$$B_{n,k} = |\mathcal{C}_n^k| = |\mathcal{C}_n^k(*,l)| + |\mathcal{C}_n^k(*,r)| = C_{n,k} + |\mathcal{C}_{n+1}^{k-1}(*,l)| = C_{n,k} + C_{n+1,k-1}.$$

□

A similar connection is true between $C_{n,k}$ and $D_{n,k}$.

Theorem 18. *We have*

$$C_{n,k} = D_{n,k} + D_{n-1,k} + D_{n-1,k+1}.$$

Proof. We know that $C_{n,k} = |\mathcal{C}_n^k(*,l)|$ and $\mathcal{C}_n^k(*,l) = \mathcal{C}_n^k(r,l)\dot{\cup}\mathcal{C}_n^k(l,l)$. It follows that

$$C_{n,k} = |\mathcal{C}_n^k(*,l)| = |\mathcal{C}_n^k(r,l)| + |\mathcal{C}_n^k(l,l)| = D_{n,k} + |\mathcal{C}_n^k(l,l)|.$$

The second term can be handled as we handled $\mathcal{C}_n^k(*,l)$ in our lemma. We present a bijection $\varphi : \mathcal{C}_n^k(l,l) \to \mathcal{C}_{n-1}^{k+1}(l,r)\dot{\cup}\mathcal{C}_{n-1}^k(r,l)$.

Let $\pi \in \mathcal{C}_n^k(l,l)$. Find the position of n (the largest left value) in π. It is the last element of one of the left blocks. If it is in the last block then simply rewrite it to $\mathbf{k+1}$. If it is not in the last block then there is a following right block R and at least one more left block. Then also rewrite it to $\mathbf{k+1}$ and at the same time move R to the end of π. The resulting permutation is $\varphi(\pi)$.

So far we did the same as we did in the proof of the lemma. The only problem is that the image is not necessarily in $\mathcal{C}_{n-1}^{k+1}(l,r)$. It is possible that the block of n is the first block of π and it consists of only one element. Then the lemma's idea leads to $\varphi(\pi)$ where the leading element is $\mathbf{k+1}$. In this very special case (n is the first element of π) we just erase n from π in order to obtain $\varphi(\pi)$.

Now it is clear that we defined a map with $\mathcal{C}_{n-1}^{k+1}(l,r)\dot{\cup}\mathcal{C}_{n-1}^k(r,l)$ as codomain. To see that it is a bijection we construct its inverse: if we have a permutation from $C_{n-1}^k(r,l)$, then the inverse puts a starting n in front of it. If we have a permutation from $C_{n-1}^{k+1}(l,r)$, then the inverse works as in our lemma.

The bijection leads to a swift conclusion of our proof:

$$C_{n,k} = D_{n,k} + |\mathcal{C}_n^k(l,l)| = D_{n,k} + |\mathcal{C}_{n-1}^k(r,l)| + |\mathcal{C}_{n-1}^{k+1}(l,r)| = D_{n,k} + D_{n-1,k} + D_{n-1,k+1}.$$

□

4. Acyclic Orientations of Bipartite Complete Graphs

The connection of poly-Bernoulli numbers to acyclic orientations of the bipartite complete graph was discovered independently in two lines of research. (An *acyclic orientation* of a graph is an assignment of direction to each edge of the graph such that there are no directed cycles.)

Cameron, Glass and Schumacher [13] investigated the problem of maximizing the number of acyclic orientations of graphs with v vertices and e edges. They conjectured that for $v = 2n$ and $e = n^2$ the extremal graph is $K_{n,n}$. Along their research they counted the acyclic orientations of $K_{n,k}$, and established a bijection between these orientations and lonesum matrices of size $n \times k$.

In [19], Section 4. the authors realized the connection of the permutations with extremal excedance sets and acyclic orientations with a unique sink. Without referring to the C-relatives of poly-Bernoulli numbers they gave an interpretation of the $C_{n,k}$ numbers in terms of acyclic orientations of complete bipartite graphs. Their proof is a specialization of general statements; we redo the version by elementary means.

We extend their results with an interpretation for $D_{n,k}$ and summarize this line of research in the next theorem. We need some notation. Let $N = \{u_1, u_2, \ldots, u_n\}$, $\widehat{N} = N \cup \{u\}$, $M = \{v_1, v_2, \ldots, v_k\}$, and $\widehat{M} = M \cup \{v\}$ be vertex sets. Let $K_{A,B}$ denote the complete bipartite graph on $A \dot\cup B$. Let $\mathcal{D}_n^k$ denote the set of acyclic orientations of $K_{N,M}$. Let $\mathcal{D}_n^{k\prime}$ denote the set of acyclic orientations of $K_{N,\widehat{M}}$, where v is the only sink (vertex without outgoing edge). Let $\mathcal{D}_n^{k\prime\prime}$ denote the set of acyclic orientations of $K_{\widehat{N},\widehat{M}}$, where u is the only source (vertex without ingoing edge) and v is the only sink.

Theorem 19. *The next three statements hold*

(i) [13]
$$|\mathcal{D}_n^k| = B_{n,k},$$

(ii) [19]
$$|\mathcal{D}_n^{k\prime}| = C_{n,k},$$

(iii)
$$|\mathcal{D}_n^{k\prime\prime}| = D_{n,k}.$$

Proof. (i): An acyclic orientation of $K_{N,M}$ can be coded by a 01 matrix B of size $n \times k$ the following way: $b_{i,j} = 0$ whenever the edge $u_i v_j$ is oriented from u_i to v_j, and $b_{i,j} = 1$ whenever the edge $u_i v_j$ is oriented from v_j to u_i. It is easy to check that the orientation is acyclic if and only if the corresponding matrix B does not contain any of the submatrix of the set L; hence, B is a lonesum matrix. This

establishes a bijection between the sets $\mathcal{D}_n^k$ and $\mathcal{L}_n^k$. The claim follows from our previous results.

(ii): Take a binary matrix that codes an orientation of a complete bipartite graph $K_{A,B}$. An all 0 column (the column of vertex $w \in B$) corresponds to the information that w is a sink. Hence, if we take an arbitrary orientation of $K_{N,\widehat{M}}$ from $\mathcal{D}_n^{k\prime}$, then its restriction to $K_{N,M}$ will be acyclic. Its coding binary matrix cannot contain an all 0 column, since an all 0 column would correspond to a second sink. Note that there is no restriction on rows. The elements of N cannot be sinks, since the edges connecting them to v are outgoing edges.

The above argument gives us a bijection between $\mathcal{D}_n^{k\prime}$ and $\mathcal{L}_n^k(c|)$ that proves our claim.

(iii): It is a straightforward extension of the previous proof. □

Some classical results about the acyclic orientations of complete bipartite graphs are as follows.

The *chromatic polynomial* of a graph G is a polynomial $\mathrm{chr}_G(q)$ such that, for a natural number k, $\mathrm{chr}_G(k)$ gives the number of good k-colorings of G. A famous result of Stanley [37] states that the number of the acyclic orientations of a graph is equal to the absolute value of the chromatic polynomial of the graph evaluated at -1. Green and Zaslavsky [22] showed that the number of acyclic orientations with a given unique sink is (up to sign) the coefficient of the linear term of the chromatic polynomial (see [21] for elementary proofs). In [22] it is also proven that the number of acyclic orientations of a graph G with a specified uv edge, such that u is the unique source and v is the unique sink, is the derivative of the chromatic polynomial evaluated at 1 (the necessary signing is taken). Again [21] presents an elementary discussion of this result.

By putting together the information quoted above, we obtain the following theorem.

Theorem 20. *The poly-Bernoulli numbers and their relatives can be expressed using the chromatic polynomial of a complete bipartite graph as follows:*

(i) ([13])
$$B_{n,k} = (-1)^{n+k} chr_{K_{n,k}}(-1),$$

(ii) ([19])
$$C_{n,k} = (-1)^{n+k}[q] chr_{K_{n,k+1}}(q),$$

(iii)
$$D_{n,k} = (-1)^{n+k}\left(\frac{d}{dq} chr_{K_{n+1,k+1}}\right)(1).$$

The chromatic polynomials of complete bipartite graphs are well-understood. We list a few results on this subject.

The exponential generating function of the chromatic polynomial of $K_{n,k}$ (See [38], Ex. 5.6):

$$\sum_{n\geq 0}\sum_{k\geq 0}\mathrm{chr}_{K_{n,k}}(q)\cdot\frac{x^n}{n!}\frac{y^k}{k!}=(e^x+e^y-1)^q.$$

Several formulas for the chromatic polynomial of complete bipartite graphs are known (See for instance [39], [19], [23]):

$$\mathrm{chr}_{K_{n,k}}(q)=\sum_{i=0}^{n}\sum_{j=0}^{k}\left\{{n \atop i}\right\}\left\{{k \atop j}\right\}(q)_{i+j},$$

where $(q)_\ell=q(q-1)(q-2)\ldots(q-\ell+1)$ is the "falling factorial", and

$$\mathrm{chr}_{K_{n,k}}(q)=\sum_{m\geq 0}\left(\sum_{i=0}^{n}\sum_{j=0}^{k}s(i+j,m)\left\{{n \atop i}\right\}\left\{{k \atop j}\right\}\right)q^m,$$

where $s(n,k)$ is the (signed) Stirling number of the first kind.

Simple arithmetic leads to the following theorem.

Theorem 21. *The following three statements hold:*

(i)

$$B_{n,k}=(-1)^{n+k}\sum_{l=0}^{n+k}\sum_{i=0}^{n}\sum_{j=0}^{k}(-1)^l s(i+j,l)\left\{{n \atop i}\right\}\left\{{k \atop j}\right\},$$

(ii)

$$C_{n,k}=(-1)^{n+k}\sum_{i=0}^{n}\sum_{j=0}^{k+1}s(i+j,1)\left\{{n \atop i}\right\}\left\{{k+1 \atop j}\right\},$$

(iii)

$$D_{n,k}=(-1)^{n+k}\sum_{l=0}^{n+k+2}\sum_{i=0}^{n+1}\sum_{j=0}^{k+1}l s(i+j,l)\left\{{n+1 \atop i}\right\}\left\{{k+1 \atop j}\right\}.$$

We note that the formula in (ii) is implicit in [19] without mentioning the poly-Bernoulli connection.

5. Algorithms for Generating the Series

In this section we recall algorithms that compute the arrays $B_{n,k}$, $C_{n,k}$ and $D_{n,k}$. We will see that in this context the array of the poly-Bernoulli relative $D_{n,k}$ arises naturally.

This line of research was initiated by the Akiyama–Tanigawa algorithm that generates the Bernoulli numbers. Let us define the array $a_{n,i}$ recursively (based on $\{a_{0,i}\}$) by the rule:

$$a_{n+1,i} = (i+1)(a_{n,i} - a_{n,i+1}).$$

Akiyama and Tanigawa [1] proved that if the initial sequence is $a_{0,i} = \frac{1}{i}$ then $a_{n,0}$ is the nth Bernoulli number. Let us denote by AT the transformation $\{a_{0,i}\} \to \{a_{n,0}\}$. The Akiyama–Tanigawa theorem says that $AT(\{1/(i+1)\}_i) = \{B_i\}_i$ (with $B_1 = \frac{1}{2}$).

Kaneko [25] showed that for any initial sequence $a_{0,i}$ the following holds

$$a_{n,0} = \sum_{i=0}^{n} (-1)^i i! \left\{ {n+1 \atop i+1} \right\} a_{0,i}.$$

Hence, we obtain the following theorem.

Theorem 22. *For all positive integers k*

(i) [25]

$$AT(\{(i+1)^k\}_i) = \{(-1)^i C_{i,k}\}_i,$$

(ii)

$$AT(\{i^k\}_i) = \{(-1)^i D_{i,k}\}_i.$$

The poly-Bernoulli numbers themselves can also be generated by such simple rules. Chen [14] changed the recursive rule in the Akiyama–Tanigawa algorithm

$$b_{n+1,i} = i b_{n,i} - (i+1) b_{n,i+1},$$

and showed that in this case

$$b_{n,0} = \sum_{i=0}^{n} (-1)^n i! \left\{ {n \atop i} \right\} b_{0,i}.$$

Let us denote by BT the transformation $\{b_{0,i}\} \to \{b_{n,0}\}$, the transformation based on the modified recursive rule. It follows that $BT(\{1/(i+1)\}_i) = \{B_i\}_i$ (with $B_1 = -\frac{1}{2}$). Furthermore, we have the following theorem.

Theorem 23. *For all positive integers k*

$$BT(\{(i+1)^k\}_i) = \{(-1)^i B_{i,k}\}_i.$$

6. Diagonal Sum of Poly-Bernoulli Numbers

The diagonal sum of poly-Bernoulli numbers as well as their relatives arise in analytical, number theoretical, and combinatorial investigations [33], [26], [2]. However,

a satisfactory formula is still missing. The diagonal sum of the poly-Bernoulli numbers

$$\sum_{n+k=N} B_{n,k}$$

are referred to in OEIS [33] A098830:

$$1, 2, 4, 10, 32, 126, 588, 3170, \ldots$$

The diagonal sum of the C-relatives are also referred to in OEIS [33] A136127:

$$1, 2, 5, 16, 63, 294, 1585, \ldots$$

The simple arithmetic relation between $B_{n,k}$ and $C_{n,k}$ of Theorem 17. implies that (except the first entry) A098830 is exactly the double of A136127.

From the combinatorial point of view, the diagonal sum enumerates sets of the combinatorial objects which we listed in this paper previously. However, there are combinatorial objects where this sum itself arises naturally: there is no reason for the division of the basic set of size N into two sets of size n and k with $n + k = N$. Here we mention some of them:

- The ascending-to-max property [20] is one of the characteristic properties of permutations that are suffix arrays of binary words. Suffix arrays play an important role in efficient searching algorithms of given patterns in a text.
- Cycles without stretching pairs [2] received attention because of their connection to a result of Sharkovsky in discrete dynamical systems. The occurrence of a stretching pair within a periodic orbit implies turbulence [17]. In [17] we find also the description of strong connections to permutations that avoid $(21 - 34)$ or $(34 - 21)$ as generalized patterns.
- The introduction of the combinatorial non-ambiguous trees [5] that are compact embeddings of binary trees in a grid, was motivated by enumeration of parallelogram polynomios. Non-ambiguous trees are actually special cases of tree-like tableaux, objects that are in one-to-one correspondence with permutation tableaux.

From the analytical results we recall here an interesting connection to the central binomial sum $CB(k)$ defined as:

$$CB(k) = \sum_{n\geq 1} \frac{n^k}{\binom{2n}{n}}.$$

Borwein and Girgensohn [10, Section 2.] showed that

$$CB(N) = P_N + Q_N \frac{\pi}{\sqrt{3}},$$

where P_N and Q_N are explicitly given rationals.

Stephan's computations suggest the following interesting conjecture [33]:

Conjecture 1. *For all* $N \geq 0$

$$\sum_{n+k=N} B_{n,k} = 3P_N.$$

Based on the explicit formula that was given in [10] we can reformulate Stephan's conjecture:

$$\sum_{n+k=N} B_{n,k} = 3P_N = (-1)^{N+1}\frac{1}{2}\sum_{j=1}^{N+1}(-1)^j j!\left\{ {N+1 \atop j} \right\}\frac{\binom{2j}{j}}{3^{j-1}}\sum_{i=0}^{j-1}\frac{3^i}{(2i+1)\binom{2i}{i}}.$$

It would be interesting to prove the conjecture or/and to find a simple expression for the diagonal sum.

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