On a problem of B. Mityagin*

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1 The problem

In connection with an uncertainty principle Boris Mityagin [2] formulated the following problem. For given $0 < p < \infty$ and $d \geq 1$, characterize those non-empty subsets A, B of $\mathbf{R}^d$ for which

$$f(\cdot + a) - f(\cdot) \in L^p(\mathbf{R}^d) \qquad \text{for all } \ a \in A, \tag{1}$$

and

$$f(\cdot)\sin\langle \cdot, b\rangle \in L^p(\mathbf{R}^d) \qquad \text{for all } \ b \in B, \tag{2}$$

imply $f \in L^p(\mathbf{R}^d)$ for any measurable function f on $\mathbf{R}^d$ (here $\langle x, b\rangle$ denotes the inner product of x and b). He showed (for $p \geq 1$) that if[1]

(i) $A = \alpha\mathbf{Z}^d$ and $B = \beta\mathbf{Z}^d$, or

(ii) $A = \{a\}$ and $B = \{b\}$ are singletons,

then (1) and (2) imply $f \in L^p(\mathbf{R}^d)$ if and only if $\alpha\beta$ is not an integer multiple of π in case (i) and $\langle a, b\rangle$ is not an integer multiple of π in case of (ii). He has also conjectured

Proposition 1 (1) *and* (2) *imply $f \in L^p(\mathbf{R}^d)$ for every measurable function f on $\mathbf{R}^d$ if and only if there are $a \in A$ and $b \in B$ such that $\langle a, b\rangle$ is not an integer multiple of π.*

This paper is devoted to the proof of this proposition. A relatively simple modification of the proofs shows that the claim is true also for $L^\infty(\mathbf{R}^d)$.

The sufficiency part of Proposition 1 easily follows from the method of [2] (which fact was mentioned in that paper), but we follow a different and shorter path.

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[1] In what follows, $\mathbf{Z}$ denotes the set of integers

2 Sufficiency in Proposition 1

Let $a \in A$ and $b \in B$ be such that $\langle a, b\rangle \notin \pi\mathbf{Z}$. If we multiply the function in (1) by $\sin\langle\cdot, b\rangle$ and add the function in (2), then we obtain $f(\cdot+a)\sin\langle\cdot, b\rangle \in L^p(\mathbf{R}^d)$, i.e. $f(\cdot)\sin\langle\cdot - a, b\rangle \in L^p(\mathbf{R}^d)$, which is the same as $f(\cdot)|\sin\langle\cdot - a, b\rangle| \in L^p(\mathbf{R}^d)$. Thus, $fh \in L^p(\mathbf{R}^d)$, where

$$h(x) = |\sin\langle x, b\rangle| + |\sin\langle x - a, b\rangle|.$$

On the line $\ell = \mathbf{R}b$ the function h (i.e. the function $h(tb)$, $t \in \mathbf{R}$) is continuous, non-zero (a zero would mean that for some t both $t\langle b, b\rangle$ and $t\langle b, b\rangle - \langle a, b\rangle$ — and hence also $\langle a, b\rangle$ — belongs to $\pi\mathbf{Z}$, which is not the case by the assumption) and periodic with period $\pi b/\langle b, b\rangle$, hence it is bounded away from 0: $h \geq \delta > 0$ on ℓ. Since h is constant on any hyperplane of $\mathbf{R}^d$ that is perpendicular to ℓ, it follows that $h \geq \delta$ everywhere, and hence $fh \in L^p(\mathbf{R}^d)$ implies $f \in L^p(\mathbf{R}^d)$.

■

3 Necessity in Proposition 1

Suppose now that

$$\langle a, b\rangle \in \pi\mathbf{Z} \qquad \text{for all } a \in A \text{ and } b \in B. \tag{3}$$

We are going to construct a measurable function $f \notin L^p(\mathbf{R}^d)$ for which (1) and (2) are true.

Let $\mathcal{A}$ be the additive group generated by A with vector addition in $\mathbf{R}^d$ as the group operation. Then

$$\langle a, b\rangle \in \pi\mathbf{Z} \qquad \text{for all } a \in \mathcal{A} \text{ and } b \in B, \tag{4}$$

is also true, hence we may replace A by $\mathcal{A}$. If $\overline{\mathcal{A}}$ is the closure of $\mathcal{A}$ in the metric of $\mathbf{R}^d$, then (4) remains true when $\mathcal{A}$ is replaced by $\overline{\mathcal{A}}$, so we may assume that $\mathcal{A}$ is a closed subgroup of $\mathbf{R}^d$. We shall need the following description of $\mathcal{A}$, which is basically known (c.f. [1, M. 4.8], [3, Theorem 4.20]) and fairly easy to prove. Since our formulation is somewhat more precise than what is in the literature, for completeness we give a proof at the and of this note.

Lemma 2 *(a) Let $\mathcal{A}$ be a closed additive subgroup of $\mathbf{R}^d$. Then there is a subspace V of $\mathbf{R}^d$ and a discrete subgroup $\mathcal{G}$ in its orthogonal complement $V^\perp$ such that $\mathcal{A} = \mathcal{G} + V$.*

(b) The discrete subgroups of $\mathbf{R}^d$ are the free groups generated by linearly independent elements.

(a) means that every $a \in \mathcal{A}$ can be uniquely written in the form $a = g + v$ where $g \in \mathcal{G}$ and $v \in V$. (b) means for the $\mathcal{G}$ in (a) that there are linearly independent elements $g_1, \dots, g_m \in \mathcal{G}$ such that every $g \in \mathcal{G}$ can be uniquely written in the form

$$g = \alpha_1(g)g_1 + \cdots + \alpha_m(g)g_m,$$

with some integers $\alpha_1(g), \dots, \alpha_m(g)$. Set

$$\alpha(g) := \max_{1 \le j \le m} |\alpha_j(g)|,$$

and

$$S_k := \{g \in \mathcal{G} \,|\, \alpha(g) = k\}.$$

Since the different elements $kg_1 + \alpha_2 g_2 + \cdots \alpha_m g_m$ with $-k \le \alpha_j \le k$ all belong to S_k, we have $|S_k| \ge (2k+1)^{m-1}$. On the other hand, every element of S_k belongs to one of the sets $\{g \,|\, \alpha_j(g) = \pm k, -k \le \alpha_i(g) \le k \text{ if } i \ne j\}$, $1 \le j \le m$. Each of these sets has $2(2k+1)^{m-1}$ elements, hence $|S_k| \le 2m(2k+1)^{m-1}$. Thus, if $P \sim Q$ means that P/Q lies in between two positive constants, then we have $|S_k| \sim (k+1)^{m-1}$ for all k. As a consequence we obtain that if $M > 0$ is any number, then for $\varepsilon \ge 0$

$$\sum_{a \in \mathcal{G}} \frac{1}{(\alpha(a) + M)^{m+\varepsilon}} < \infty \quad \Leftrightarrow \quad \varepsilon > 0. \tag{5}$$

Indeed, this is immediate since

$$\begin{aligned} \sum_{a \in \mathcal{G}} \frac{1}{(\alpha(a)+M)^{m+\varepsilon}} &= \sum_{k=0}^{\infty} \sum_{a \in S_k} \frac{1}{(\alpha(a)+M)^{m+\varepsilon}} = \sum_{k=0}^{\infty} \frac{|S_k|}{(k+M)^{m+\varepsilon}} \\ &\sim \sum_{k=0}^{\infty} \frac{(k+1)^{m-1}}{(k+M)^{m+\varepsilon}}, \end{aligned}$$

and it is clear that the last sum diverges (terms are $\sim 1/k$) if $\varepsilon = 0$, and converges (terms are $\sim 1/k^{1+\varepsilon}$) if $\varepsilon > 0$.

In the proof of the necessity we distinguish two cases.

Case I: $\mathcal{A}$ is discrete. Thus, in this case $V = \{0\}$ and $\mathcal{A} = \mathcal{G}$. Since $\mathcal{A}$ is discrete, there is an M such that the distance in between different elements of $\mathcal{A}$ is at least $2/M^{1/d}$ (just note that if there were different elements arbitrarily close to each other, then their difference would be non-zero and arbitrarily close to 0, contradicting the discrete character of $\mathcal{A}$).

Assume first that the number m in the description of $\mathcal{G}$ is bigger than 0. For $a \in \mathcal{A}$ let $\mathcal{B}_a$ be the (closed) ball of radius $1/(\alpha(a) + M)^{m/d}$ with center at a, and set $f = \chi_{\cup_{a \in \mathcal{A}} \mathcal{B}_a}$, where χ_E denotes the characteristic function of the set E. Since the balls $\mathcal{B}_a$ are disjoint by the choice of M, and the d-dimensional volume of a ball of radius r is $\theta_d r^d$ with some number θ_d, it follows that the L^1 norm of f^p is

$$\theta_d \sum_{a \in \mathcal{A}} \frac{1}{(\alpha(a) + M)^m} = \infty$$

by (5), so $f \notin L^p(\mathbf{R}^d)$. On the other hand, below we show that (1) and (2) are true, and that will complete the proof of the necessity in the case when $\mathcal{A}$ is discrete and $m \geq 1$.

It is sufficient to prove (1) for the generators g_j, $j = 1, \ldots, m$. Choose such a g_j, and consider the set

$$F_j = \{x \,|\, f(x + g_j) - f(x) \neq 0\}.$$

Since $f(x+g_j)-f(x)$ takes only the values $0, \pm 1$, if we show that $\text{meas}(F_j) < \infty$, then (1) follows. But $x \in F_j$ means that either $x \in \mathcal{B}_{a_0}$ for some $a_0 \in \mathcal{A}$ and $x + g_j \notin \cup_{a\in\mathcal{A}}\mathcal{B}_a$, or the other way around (i.e. $x + g_j \in \mathcal{B}_{a_0}$ and $x \notin \cup_{a\in\mathcal{A}}\mathcal{B}_a$). These two cases are similar (just replace x by $x + g_j$ and g_j by $-g_j$), so consider the first one. Let $B_r(z)$ denote the (closed) ball about z and of radius r. Since $x + g_j \notin \cup_{a\in\mathcal{A}}\mathcal{B}_a$, we have in particular

$$x + g_j \notin \mathcal{B}_{a_0+g_j} = B_{(\alpha(a_0+g_j)+M)^{-m/d}}(a_0 + g_j),$$

which is the same as

$$x \notin B_{(\alpha(a_0+g_j)+M)^{-m/d}}(a_0).$$

Therefore, by the assumption

$$x \in B_{(\alpha(a_0)+M)^{-m/d}}(a_0) \setminus B_{(\alpha(a_0+g_j)+M)^{-m/d}}(a_0). \tag{6}$$

This is possible only if $\alpha(a_0 + g_j) > \alpha(a_0)$. But in any case, the definition of the function α shows that $\alpha(a_0 + g_j) \leq \alpha(a_0) + 1$, so we must have $\alpha(a_0 + g_j) = \alpha(a_0) + 1$. But then from (6) it follows that

$$\begin{aligned} \text{meas}(F_j \cap \mathcal{B}_{a_0}) &\leq 2 \text{ meas} \left(B_{(\alpha(a_0)+M)^{-m/d}}(a_0 + g_j) \setminus B_{(\alpha(a_0+g_j)+M)^{-m/d}}(a_0 + g_j) \right) \\ &= 2\theta_d \left(\frac{1}{(\alpha(a_0) + M)^m} - \frac{1}{(\alpha(a_0) + 1 + M)^m} \right) \sim \frac{1}{(\alpha(a_0) + M)^{m+1}}, \end{aligned}$$

and so

$$\text{meas}(F_j) = \sum_{a_0\in\mathcal{A}} \text{meas}(F_j \cap \mathcal{B}_{a_0}) < \infty$$

in view of (5). This proves (1).

Now consider property (2). Let $b \in B$. Since $\langle a, b\rangle \equiv 0 \pmod{\pi}$ for all $a \in \mathcal{A}$, it follows that if $x \in \mathcal{B}_a$, then (in what follows $|x|$ denotes the Euclidean norm of $x \in \mathbf{R}^d$)

$$|\sin\langle x, b\rangle| = |\sin\langle x - a, b\rangle| \leq |x - a||b| \leq \frac{|b|}{(\alpha(a) + M)^{m/d}},$$

and so

$$\int_{\mathcal{B}_a} |f(x) \sin\langle x, b\rangle|^p dx \leq \left(\frac{|b|}{(\alpha(a) + M)^{m/d}} \right)^p \text{meas}(\mathcal{B}_a) = |b|^p \frac{\theta_d}{(\alpha(a) + M)^{m+mp/d}}.$$

Therefore, (5) implies

$$\int |f(x)\sin\langle x,b\rangle|^p dx = \sum_{a\in\mathcal{A}} \int_{\mathcal{B}_a} |f(x)\sin\langle x,b\rangle|^p dx = \sum_{a\in\mathcal{A}} |b|^p \frac{\theta_d}{(\alpha(a)+M)^{m+mp/d}} < \infty.$$

This is property (2), and the proof is complete when $\mathcal{A}$ is discrete and $m \geq 1$.

If $\mathcal{A}$ is discrete but $m = 0$, then $\mathcal{A} = A = \{0\}$, so (1) is automatic for all f, and to get the necessity just set $f(x) = |x|^{-d/p}(1+|x|)^{-2}$ which function is not in $L^p(\mathbf{R}^d)$, but $f(\cdot)|\cdot| \in L^p(\mathbf{R}^d)$ (which relation is needed only around 0) implying (2).

Case II: $\mathcal{A}$ is not discrete. In this case, $V \neq \{0\}$. Let $l \geq 1$ be the dimension of V, and assume first again that $\mathcal{G} \neq \{0\}$, i.e. $m \geq 1$. Since $\mathcal{G}$ is discrete, there is an $M > 0$ such that different elements of $\mathcal{G}$ are of distance $> 2/M^{(m+l)/(d-l)}$. This implies that any two elements of $g + V$ and $g' + V$ are of distance $> 2/M^{(m+l)/(d-l)}$ if $g, g' \in \mathcal{G}$ are different (note that $\mathcal{G}$ lies in $V^\perp$).

Let $\mathcal{D}$ be the (closed) unit ball in $V^\perp$. It is of dimension $d-l > 0$ (note that V cannot be the whole $\mathbf{R}^d$ because $m \geq 1$), and for a $y \in V$ and $g \in \mathcal{G}$ let

$$\mathcal{D}_{y,g} = y + g + \mathcal{D}\cdot(|y| + \alpha(g) + M)^{-(m+l)/(d-l)},$$

which is a $d-l$ dimensional ball about $g+y$ of radius $(|y|+\alpha(g)+M)^{-(m+l)/(d-l)}$. Set

$$E_g = \cup_{y\in V}\mathcal{D}_{y,g}$$

and $f = \chi_{\cup_{g\in\mathcal{G}}E_g}$. According to what we have just said, the different E_g's are disjoint (since any element of E_g is of distance $\leq 1/(\alpha(g)+M)^{(m+l)/(d-l)} \leq 1/M^{(m+l)/(d-l)}$ from $g+V$). It is easy to see that each E_g is closed, so f is measurable. Using Fubini's theorem we obtain that

$$\mathrm{meas}(E_g) = \int_V \theta_{d-l}\frac{1}{(|y|+\alpha(g)+M)^{m+l}}dy \sim \frac{1}{(\alpha(g)+M)^m}, \qquad (7)$$

where we used that for $\tau \geq 0$

$$\int_V \frac{1}{(|y|+L)^{m+l+\tau}}dy \sim \frac{1}{L^{m+\tau}} \qquad (8)$$

uniformly in $L > 0$. Indeed, this is immediate if we make the substitution $y = Ly'$ in the integral.

In view of (7) and (5)

$$\mathrm{meas}\left(\cup_{g\in\mathcal{G}}E_g\right) \sim \sum_{g\in\mathcal{G}} \frac{1}{(\alpha(g)+M)^m} = \infty,$$

and hence $f \notin L^p(\mathbf{R}^d)$. To complete the proof we shall show that, on the other hand, f satisfies both (1) and (2).

It is enough to prove (1) for all $a = v$, $v \in V$ and for all generators $a = g_j$ of $\mathcal{G}$. This second one is similar to what we did in the discrete case. Indeed, let again

$$F_j = \{x \mid f(x + g_j) - f(x) \neq 0\},$$

and it is sufficient to show that $\text{meas}(F_j) < \infty$. Now $x \in F_j$ means that either $x \in \mathcal{D}_{y,a_0}$ for some $y \in V$ and $a_0 \in \mathcal{G}$ and $x + g_j \notin \cup_{g\in\mathcal{G}} E_g$, or the other way around, and we may consider the first case. Then

$$x \in y + a_0 + \mathcal{D} \cdot (|y| + \alpha(a_0) + M)^{-(m+l)/(d-l)}$$

but

$$x + g_j \notin y + a_0 + g_j + \mathcal{D} \cdot (|y| + \alpha(a_0 + g_j) + M)^{-(m+l)/(d-l)},$$

i.e.

$$x \notin y + a_0 + \mathcal{D} \cdot (|y| + \alpha(a_0 + g_j) + M)^{-(m+l)/(d-l)},$$

and so

$$x \in y + a_0 + \left(\frac{\mathcal{D}}{(|y| + \alpha(a_0) + M)^{(m+l)/(d-l)}} \setminus \frac{\mathcal{D}}{(|y| + \alpha(a_0 + g_j) + M)^{(m+l)/(d-l)}} \right). \tag{9}$$

As in the discrete case this is possible only if $\alpha(a_0 + g_j) = \alpha(a_0) + 1$, and then it follows that the $(d - l)$-dimensional measure of $F_j \cap (E_{a_0} \cap (y + V^\perp))$ is at most twice the difference

$$\frac{\theta_{d-l}}{(|y| + \alpha(a_0) + M)^{m+l}} - \frac{\theta_{d-l}}{(|y| + \alpha(a_0) + 1 + M)^{m+l}} \sim \frac{1}{(|y| + \alpha(a_0) + M)^{m+l+1}}.$$

If we integrate this with respect to $y \in V$, then we obtain from (8) that the measure of $F_j \cap E_{a_0}$ is at most a constant times $(\alpha(a_0) + M)^{-(m+1)}$, and hence

$$\text{meas}(F_j) = \sum_{a_0 \in \mathcal{G}} \text{meas}(F_j \cap E_{a_0}) \leq C \sum_{a_0 \in \mathcal{G}} \frac{1}{(\alpha(a_0) + M)^{m+1}} < \infty,$$

where we used again (5).

Consider now (1) for $a = v \in V$. This time set

$$F_v^* = \{x \mid f(x + v) - f(x) \neq 0\}.$$

Now $x \in F_v^*$ means that either $x \in \mathcal{D}_{y,a_0}$ for some $y \in V$ and $a_0 \in \mathcal{G}$ and $x + v \notin \cup_{g\in\mathcal{G}} E_g$, or the other way around, and consider again the first case. Then

$$x \in y + a_0 + \mathcal{D} \cdot (|y| + \alpha(a_0) + M)^{-(m+l)/(d-l)}$$

but

$$x + v \notin y + v + a_0 + \mathcal{D} \cdot (|y + v| + \alpha(a_0) + M)^{-(m+l)/(d-l)},$$

i.e.

$$x \notin y + a_0 + \mathcal{D} \cdot (|y + v| + \alpha(a_0) + M)^{-(m+l)/(d-l)}.$$

Hence

$$x \in y+a_0+\left(\frac{\mathcal{D}}{(|y| + \alpha(a_0) + M)^{(m+l)/(d-l)}} \setminus \frac{\mathcal{D}}{(|y| + |v| + \alpha(a_0) + M)^{(m+l)/(d-l)}} \right).$$

It follows that the $(d-l)$-dimensional measure of $F_v^* \cap (E_{a_0} \cap (y + V^\perp))$ is at most twice the difference

$$\frac{\theta_{d-l}}{(|y| + \alpha(a_0) + M)^{m+l}} - \frac{\theta_{d-l}}{(|y| + |v| + \alpha(a_0) + M)^{m+l}} \sim \frac{1}{(|y| + \alpha(a_0) + M)^{m+l+1}}$$

(in this very last step the $\sim$ depends on $|v|$ but not on y or a_0). If we integrate this with respect to $y \in V$, then we obtain from (8) that the measure of $F_v^* \cap E_{a_0}$ is at most a constant times $(\alpha(a_0) + M)^{-(m+1)}$, and hence

$$\text{meas}(F_v^*) = \sum_{a_0 \in \mathcal{G}} \text{meas}(F_v^* \cap E_{a_0}) \leq C \sum_{a_0 \in \mathcal{G}} \frac{1}{(\alpha(a_0) + M)^{m+1}} < \infty,$$

because of (5). This finishes the proof of (1).

Next, consider property (2). Let $b \in B$. Since $\langle a, b \rangle \equiv 0 \pmod{\pi}$ for all $a \in \mathcal{A}$, it follows that if $x \in \mathcal{B}_{y,g}$ then

$$|\sin\langle x, b\rangle| = |\sin\langle x - g - y, b\rangle| \leq |x - g - y||b| \leq \frac{|b|}{(|y| + \alpha(a) + M)^{(m+l)/(d-l)}},$$

and so

$$\begin{aligned} \int_{\mathcal{B}_{y,g}} |f(x) \sin\langle x, b\rangle|^p dx &\leq \left(\frac{|b|}{(|y| + \alpha(a) + M)^{(m+l)/(d-l)}} \right)^p \theta_{d-l}(\text{radius of } \mathcal{B}_{y,g})^{d-l} \\ &= |b|^p \theta_{d-l} \frac{1}{(|y| + \alpha(a) + M)^{m+l+(m+l)p/(d-l)}}. \end{aligned}$$

If we integrate this for $y \in V$ then (8) implies

$$\begin{aligned} \int_{E_g} |f(x) \sin\langle x, b\rangle|^p dx &\leq \int_V |b|^p \theta_{d-l} \frac{1}{(|y| + \alpha(a) + M)^{m+l+(m+l)p/(d-l)}} dy \\ &\sim \frac{1}{(|y| + \alpha(a) + M)^{m+(m+l)p/(d-l)}}. \end{aligned}$$

Therefore, we obtain from (5)

$$\int |f(x) \sin\langle x, b\rangle|^p dx = \sum_{g \in \mathcal{G}} \int_{E_g} |f(x) \sin\langle x, b\rangle|^p dx \sim \sum_{g \in \mathcal{G}} \frac{1}{(\alpha(a) + M)^{m+(m+l)p/(d-l)}} < \infty,$$

and the proof of the necessity is complete when $m \geq 1$.

If $m = 0$ (i.e. $\mathcal{G} = \{0\}$) but $V \neq \mathbf{R}^d$, then do the preceding proof with $m = 0$ with the modification that now instead of (8) we use

$$\int_V \frac{1}{(|y| + L)^l} dy = \infty.$$

However, in the $m = 0$ case it is now possible that $V = \mathbf{R}^d$. In that case necessarily $B = \{0\}$, so (2) is automatic, and to have the necessity just pick a function f on $\mathbf{R}^d$ which is not in L^p but for which (1) holds for all $a \in \mathbf{R}^d$ (for example, set $f(x) = (|x| + 1)^{-d/p}$.)

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4 Proof of Lemma 2

For part (b) see [3, Theorem 4.20]. To prove part (a), let $\mathcal{A} \subset \mathbf{R}^d$ be the closed group in question. Let $V \subset \mathbf{R}^d$ be the largest subspace of $\mathbf{R}^d$ that lies in $\mathcal{A}$ (since the sum of two subspaces lying in $\mathcal{A}$ also lies in $\mathcal{A}$, there is such a largest subspace), and let $V^\perp$ be the orthogonal complement of V. We claim that there is a $\delta > 0$ such that all $a \in \mathcal{A} \setminus V$ lies of distance $\geq \delta$ from V. Indeed, if this is not the case, then for every n there are $a_n \in \mathcal{A}$ that lie outside V such that their distance from V is $< 1/n$. Let $v_n \in V$ be the closest element of V to a_n. Then $a_n - v_n \in V^\perp$. By compactness, the sequence $\{(a_n - v_n)/|a_n - v_n|\}$ has a convergent subsequence, and we may assume that $(a_n - v_n)/|a_n - v_n| \to u$. Then u is a unit vector lying in $V^\perp$. If $\lambda > 0$, then $(a_n - v_n)[\lambda/|a_n - v_n|] \to \lambda u$, where $[\cdot]$ denotes integral part, and since each $(a_n - v_n)[\lambda/|a_n - v_n|]$ belongs to $\mathcal{A}$, we obtain that $\lambda u \in \mathcal{A}$ for all $\lambda > 0$, and hence for all $\lambda \in \mathbf{R}$. But this means that all vectors $v + \lambda u$, $v \in V$, $\lambda \in \mathbf{R}$, lie in $\mathcal{A}$, which is impossible by the maximality of V. As a corollary it follows that $\mathcal{G} := \mathcal{A} \cap V^\perp$ is a discrete group (if we had different elements $a, a' \in \mathcal{A} \cap V^\perp$ arbitrarily close to each other, then their difference $a - a'$ would be in $V^\perp$ and hence would lie outside V, but would lie close to zero, and hence to V, which is not possible).

Every $a \in \mathcal{A}$ has a unique representation $a = a_{V^\perp} + a_V$ with $a_{V^\perp} \in V^\perp$ and $a_V \in V$. Since $a_V \in V \subset \mathcal{A}$, it follows that $a_{V^\perp} \in \mathcal{A}$. Therefore, $\mathcal{G} := \{a_{V^\perp}\} = \mathcal{A} \cap V^\perp$, so this is a subgroup, and part (a) follows.

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