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Developing and Assessing Mathematical Reasoning

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Introduction

Mathematics is one of the oldest scientific disciplines and offers valid content for school curricula. There are obvious possibilities for the application of its basics in everyday life, but a great majority of mathematical knowledge is taught in the hope that learning mathematics, besides improving reasoning and cultivating the mind in general, can provide students with systematic ways of approaching a variety of problems and with tools for analyzing and modeling situations and events in the physical, biological and social sciences. The power of mathematics as a tool for understanding the world was proclaimed by Galileo in unambiguous terms when he wrote that this great book of the universe, which stands continually open to our gaze, cannot be understood unless one first learns to comprehend the language and to read the alphabet in which it is composed: the language of mathematics (in Sobel, 1999).

In contrast to mathematics, scientific research into teaching and learning mathematics is a relatively young discipline; it is about a century old. The questions considered worth investigating and the research methods used to answer these questions changed over time but one question remains central in developmental psychology and education: Does learning mathematics improve reasoning or is mathematics learning only open to those who have attained an appropriate level of reasoning to begin with? Improving general cognitive abilities is especially important in a rapidly changing social environment; thus an answer to this question is urgent.

Modern developmental psychology has seen the rapprochement of two seemingly divergent theories that seek to explain cognitive development. On the one hand, Piaget and his colleagues analyzed the forms of reasoning that seem to characterize children's thinking as they grow up, focusing on the child's problem solving strategies (i.e., their actions and inferences) and justifications for these strategies (Inhelder & Piaget, 1958; Piaget & Inhelder, 1974, 1975, 1976). On the other hand, Vygostky paved the way for a deeper understanding of how cultural systems of signs (such as number systems, graphing and algebra) allow students to record their own thoughts externally, and then think and talk about these external signs, making them into objects and tools for thinking (Vygostky, 1978).

A simple example can illustrate this point. When anyone asks us for the time, we immediately look at our watches. In everyday life and in science, we think about time in ways that are influenced by the mathematical relations embodied in clocks and watches. We say "a day has 24 hours" because we measure time in hours; the ratio between 1 day and hours is 24:1; the ratio hours to minutes is 60:1, and the ratio minutes to seconds is 60:1. We represent time through this cultural tool – the watch – and the mathematical relations embodied in the watch, which allow us to describe the duration of a day. This cultural tool enables us to make fine distinctions between different times and also structures the way we think about time. Without it, we could not make an appointment with a friend, for example at 11 o'clock, and then say to the friend: "I'm sorry, I am 10 minutes late". Our perception of time is not that precise that we would be able to know exactly the time-point in the day that corresponds to 11 o'clock and to tell the difference between 11 and 11 : 10. This is the Vygotskian side of the story.

The Piagetian story comes into play when we think about what children need to understand in order to learn to read the watch and to compare different points in time. Numbers on the face of the watch have two meanings: they show the hours and the minutes. In order to read the minutes, children need to be able to relate 1 and 5, 2 and 10, 3 and 15 etc. and in order to find the interval, for example, between 1 : 35 and 2 : 15, they need to know that the hour has 60 minutes, and add the minutes up to 2 o'clock to the minutes after 2 o'clock. All this thinking has to be applied to the tool in order for children to learn to use it. We do not dwell on further examples here: it seems quite clear that learning to use a watch requires an understanding of the relations between minutes, hours and the numbers on the face of the watch. Research shows that this can still be challenging for 8-year-olds (Magina & Hoyles, 1997).

In this chapter we focus alternatively on the forms of reasoning that are necessary insights for learning mathematics and on the learning of conventional mathematical signs that enable and structure reasoning. The chapter is divided in three sections: whole numbers, rational numbers and solving problems in the mathematics classroom. In each of these sections we attempt to identify issues related to the psychological principles of learning mathematics and its cultural tools. The chapter is written with a focus on mathematics learning in primary school, and considers research with children mostly in the age range 5 to 12. There is no attempt to cover research about older students and no assumption that the issues raised here will suffice to understand further mathematics learning.

Those reasoning processes which are at the center of mathematics education are shaped by pre-school experiences and are influenced by outside school activities as well. Reasoning abilities developed in mathematics are applied to learning other school subjects while learning experiences in other areas may advance the development of mathematical reasoning. Well designed science education activities, for example, may stimulate those thinking abilities which are essential in mathematics too, first of all by providing experiential basis and practicing in the field. However, there are issues in cognitive development, where mathematics education plays a dominant role, such as reasoning with quantities and measures, using mathematical symbols etc. In this chapter we focus on these issues discussing in more details their critical position in further and broader mathematical development. At the same time, we acknowledge the importance of the role that mathematics education plays in promoting several further reasoning processes, but in this chapter we deal with them only in brief.

Mathematics Education and Cognitive Development

Whole Numbers

The aim of learning numbers in the initial years of primary school is to provide children with symbols for thinking and speaking about quantities. Later on in school students may be asked to explore the concept of number in a more abstract way and to analyze number sequences that are not representations of quantities, but throughout most of the primary school years numbers will be used to represent quantities and relations between them.

Quantities and numbers are not the same. Thompson (1993) suggested that "a person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it. Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them. You can think of your height, another person's height, and the amount by which one of you is taller than the other without having to know the actual values" (pp 165–166). You can also know, without using numbers, that if you are taller than your friend Rick, and Rick is taller than his friend Paula, you are taller than Paula. You are certain of this even if you have never met Paula. So we can think about relations between quantities without having a number to represent them. But when we can represent them with numbers, we can know more: If you know that you are 4 cm taller than Rick and that Rick is 2 cm taller than Paula, you know that the difference between yours and Paula's height is 6 cm.

In the initial years in primary school, children learn about numbers as tools for thinking and speaking about quantities. The emphasis we place here on numbers as representations rests on the significance of number systems as tools for thinking. We can't record quantities or communicate with others about them if we do not have a number system to represent quantities. A system for representing quantities allows us to make distinctions that we may not be able to make without the system. For example, we may not be able to tell just from looking the difference between 15 and 17 buttons, but we have no problem in doing so if we count them. Or we may not be able to tell whether a cupboard we want to buy, which we see in a shop, fits into a space in our house, but we will know if we measure the cupboard and the space where we want it to fit. Systems of representation of quantities allow us to make distinctions we cannot make perceptually and make comparisons between quantities across time and space. They enable and structure our thinking: we think with the numbers we use in measuring.

Thus, there are two crucial insights that children need to attain in order to understand whole numbers. First, they must realize that their knowledge of numbers and of quantities should be connected. Second, they must understand how the number system works.

Piaget, and subsequently many other researchers, explored several ideas that children should have about the connections between quantities and their numerical representation. Children should know, for example, that:

(1) if two quantities are equivalent, they should be represented by the same number
(2) if two quantities are represented by the same number, they are equivalent
(3) if some items are added to a set, the number that represents the set should change and should be a larger number
(4) if some items are taken away from the set, the number should change and be a smaller number
(5) if the same number of items is added to and then subtracted from the set, the quantity and the number of items in the set do not change (i.e. they should understand the inverse relation between addition and subtraction).

These insights into the relationship between number and quantity do not seem to be available to children younger than about four or five years (see Ginsburg, Klein, & Starkey, 1998, for a review of the first four points; see Nunes, Bryant, Hallett, Bell, & Evans, 2009, for a review regarding the last point). There is research that suggests that young children, even babies, can see that, if you add one item to a set of one, you should have two items, but there is no evidence to indicate that babies know that there is a connection between quantities and numerical symbols. All testing in the studies with babies is perceptual, and thus they tell us nothing about knowing that a set represented by the number 1 should no longer be represented by 1 after you add items to it. The difference between perceptual judgments and the use of symbols is at the heart of understanding mathematical learning and reasoning.

These five insights regarding the connection between quantities and numbers are necessary (but not sufficient, as it will be argued later) for understanding whole numbers but they do not have the same level of difficulty.

The first four are considerably simpler than the last one, which we refer to as the inverse relation between addition and subtraction.

The difficulty of understanding the inverse relation between addition and subtraction results from the need to coordinate two operations, addition and subtraction, with each other and to understand how this coordination affects number; it does not result from the amount of information that the children need to consider in order to answer the question. Bryant (2007) demonstrated this in a study where children were asked to consider the same amount of information about sets; some problems involved the inverse relation between addition and subtraction whereas others did not. In the inverse problems, the same number of items were added to and subtracted from a single set. In the problems that did not involve inversion, the same number of items was added to one set and subtracted from an equivalent set. Some children were able to realize that the originally equivalent sets differed after items were added to one and subtracted from the other but nevertheless did not succeed in the inverse relation items, which involved operations on the same set.

If understanding the connection between quantities and their numerical representation really is important, there should be a relationship between children's insights into these connections and their learning of mathematics. Children who already realize how quantity and number are related when they start school should have an advantage in mathematics learning in comparison to those who did not attain these insights. Two studies carried out by different research teams (Nunes, Bryant, Evans, Bell, Gardner, Gardner, & Carraher, 2007; Stern, 2005) show that children's understanding of the inverse relation between addition and subtraction predicts their mathematical achievement at a later time, even after controlling for general cognitive factors such as intelligence and working memory.

Children's understanding of the inverse relation develops over time. Children are at first able to realize that there is an inverse relation between addition and subtraction if the problems are presented to them with the support of quantities (either visually available or imagined); later, they also seem to understand this when asked about numbers, with no reference to quantities. If asked what is 34 plus 29 minus 29, they know they do not need to compute the sums: they know that the answer is 34. They may be able to also know the answer to 34 + 29 – 28 without calculating, but this is a more difficult question.

In summary, in order to understand whole numbers, children must realize that there are specific connections between quantities and numbers. At about age 4 to 5, children understand that, when two sets are equivalent, if they count one set they know how many items are in the other without having to count. At about age 6, they are able to understand also the inverse relation between addition and subtraction, and know that the number does not change if the same number of items is added to and subtracted from a set. This insight is a strong predictor of later mathematics achievement.

The insights we described in the previous paragraphs are about the logical relations between quantities and numbers but this is not all one should consider when analyzing whole numbers. One must ask also: when numbers are represented using a base-ten system, what demands does the nature of this representation place on the learner's cognitive skills? The base-ten system places two demands on the learner's cognition: the learner must also have some insight into additive composition and into multiplicative relations.

Additive relations require thinking about part-whole relations. In order to understand what 25, for example, means, the learner should understand that the two parts, 20 and 5, together are exactly as much as the whole, 25. In more general terms, the learner must understand additive composition of numbers, which means that any number can be formed by the sum of two other numbers.

The multiplicative relations in the base ten system have to do with the way the number labels and the place value system work. When we write numbers, the place where the digit is indicates an implicit multiplication: if the digit is the last one on the right, it is multiplied by 1, the second to the left is multiplied by 10, and the third to the left is multiplied by 100 and so on.

Young children's understanding of these additive and multiplicative relations in the number system may be subtle and implicit so we need specific tasks to assess this knowledge. We have created tasks that seem to assess additive composition and early multiplicative reasoning, which can be used to predict children's mathematics achievement. Additive composition is assessed by our "Shop Task". We ask children to pretend to buy items in a shop; they are given coins of different values to buy the items. If they want to buy, for example, a toy car that costs 9 cents, and they have one 5-cent coin and six 1-cent coins, they need to combine the 5-cent coin with four 1-cent coins. Children who do not understand additive composition think that they do not have exact change to pay for the toy: they say that they have five and six cents but their money does not allow them to "make" 9 cents. About two thirds of children aged 6 years pass this question. This question is highly predictive of mathematics achievement later on in primary school (Nunes et al., 2007; Nunes, Bryant, Barros, & Sylva, 2011).

We assess young children's understanding of multiplicative relations by asking them to solve multiplication and division problems using objects. For example, we show them a row of four houses and invite them to imagine that inside each house live three rabbits. We then ask them how many rabbits live in these houses. Children who have some early understanding of multiplicative relations in action simply point three times to each house and "count the rabbits" as they point to the houses. Young children's ability to pass items such as this helps predict their mathematical achievement later on (Nunes et al., 2007; Nunes, Bryant, Barros, & Sylva, 2011).

In summary, children must attain two sorts of insights in order to understand whole numbers. They need to understand the connections between quantities and numbers, and they need to understand the principles implicit in the number system that we use to represent whole numbers, which is a base-ten system. Research indicates that children who attain these insights at the beginning of primary school show higher levels of mathematical achievement later on, when the children are 8, 11 and 14 years (Nunes, Bryant, Barros, & Sylva, 2011). So, early assessments of mathematics should include items that measure such insights in order to help teachers make decisions about what to teach to their children.

Rational Numbers

Rational numbers are needed to express parts of the whole. These quantities appear in measurement and quotient situations. In a measurement situation, for example, if you are measuring sugar with a cup and the quantity you have is less than a cup, you might describe it as a third of a cup – or, with numerical symbols, 1/3. In a quotient situation, for example, you might be sharing one chocolate among three children; each child receives the result of dividing 1 by 3, or 1/3. These two situations in which fractions are used have in common the fact that, in order to speak of fractions, a division in equal parts has to take place. Fractions, thus, are numbers that result from division, rather than from counting, as whole numbers do. (Here we always mean positive parts of positive wholes.)

A division has three terms

(1) a dividend, which is the quantity being divided
(2) a divisor, which is the number of parts into which the quantity is divided
(3) a quotient, which is the result of the division and the value represented by the fraction.

In order to understand fractions as representations of quantities, children need to understand the connections between these numbers and the quantities that they represent. Fractions differ from whole numbers in many ways: we consider three basic differences here that must be mastered by students if they are to understand these numbers.

(1) A term within a fraction is given meaning by its relation to the other term: thus by knowing only the numerator we can not tell the quantity represented by the fraction.
(2) The same fraction might represent different quantities when the fraction itself bears a relation to a whole. So ½ of 8 and ½ of 12 are not equivalent although they are expressed by the same fraction.
(3) Different fractions might represent the same quantity: ½ and 2/4 of the same pie represent the same quantity; this is treated in the mathematics classroom as the study of equivalent fractions.

Many students do not seem to understand at first that the numbers in a fraction represent relations between quantities (Vamvakoussi & Vosniadou, 2004); it takes some time for this understanding to develop, at least under the present conditions of instruction. We explore below some of the ways in which this aspect of understanding fractions has been investigated.

One relation that students must understand is that, the greater the dividend, the greater the quotient, if the divisor remains the same. In part-whole situations, the dividend is the whole, which is not explicit in the fractional numerical representation; when we say 1/3 cup, the quantity in a cup is what is being divided. It may be easy to understand that 1/3 of a small cup and 1/3 of a large cup will not be the same quantity. But perhaps it is not as easy for students to understand that the quantity represented by the symbol 1/3 may not always be the same because the quantity being divided may not be the same.

We know of no studies that included a question about whether the same fraction may represent different numbers (when expressing fractions of different wholes) but Hart, Brown, Kerslake, Kücherman, and Ruddock (1985) included in their large scale study of students' understanding of fractions a question that investigates students' understanding of the connection between fraction symbols and quantities. They told students that Mary spent ¼ of her pocket money and John spent ½ of his pocket money, and then asked: is it possible that Mary spent more money than John? If students understand that the size of the whole matters, they should say that it is indeed possible that Mary spent more money, although ½ is more than ¼ if the quantities come from the same whole. However, 42% of the 11–12 year olds and 34% of the 12–13 year olds said that it is not possible; they justified their answer by indicating that ½ is always more than ¼. So, it is not obvious to students in this age range that the same fraction might not represent equivalent quantities.

Understanding the equivalence of fractions – that is, that different fractions may represent the same quantity – is crucial for connecting quantities with fractional symbols and also for adding and subtracting fractions. Research suggests that fraction equivalence is not easy for many students (e.g. Behr, Wachsmuth, Post, & Lesh, 1984; Kerslake, 1986) and that this is not an all-or-nothing insight: students might attain this insight in one type of situation but not in another. We (Nunes, Bryant, Pretzlik, Bell, Evans, & Wade, 2007) investigated students' (age range 8 to 10 years) understanding of the equivalence of fractional quantities in the context of part-whole and quotient situations, both presented with the support of drawings. The problem in the part-whole situation was: Peter and Alan were given chocolate bars of the same size, which were too large to be eaten in one day. Peter broke his chocolate in 8 equal parts and ate 4; Alan broke his chocolate in 4 equal parts and ate 2. The students were asked whether the boys ate the same amount of chocolate. The rate of correct responses to this problem was 31%. The problem in the quotient situation was: a group of 4 girls is sharing equally one cake and a group of 8 boys is sharing equally two cakes which are identical to the girls' cake. The students were asked whether, after the division, each girl would eat the same amount of cake as each boy. The rate of correct responses in this situation was 73%. Thus, understanding the equivalence between fractional quantities seems to happen in different steps: quotient situations lead to significantly better performance.

The difference in students' performance between these two situations surprises many teachers but it is important to remember that problems that seem very similar to a mathematician can be perceived as completely different by students (Vergnaud, 1979). Developmental psychologists test whether children perceive different objects as instances of the same category by teaching them to name one object and asking them to name the second one, without any instruction. If the children generalize the name learned for the first object to the second, one can infer that they see both as instances of the same category.

This approach has been used in the analysis of fractions in two studies (Nunes, Campos & Bryant, 2011; Mamede, 2007). In these studies, two groups of students who had not yet received instruction about fractions in school were introduced to the use of fractional representation in an experiment. The students were randomly assigned to one condition of instruction: they either learned to use fraction symbols to represent part-whole relations or to represent quantities in quotient situations. Both groups of students progressed in the use of fractions symbols from pre- to post-test and made significantly more progress than a control group, but this progress was specific to the situation in which they received instruction. Students who learned to use fractions for part-whole relations could not use fractions to represent quotient situations, and vice-versa. So, children do not immediately see that they can use fractions to represent part-whole and quotient situations: they do not generalize the use of these symbols from one situation to the other. This finding should caution researchers about drawing general conclusions about students' knowledge of fractions if they have analyzed the students' performance in only one type of situation.

Finally, putting fractions in order of magnitude involves understanding the relationship between the divisor and the quotient in a division, or between the denominator and the quantity represented in a fraction: if the numerator is constant, the larger the denominator, the smaller the quantity represented. Children seem understand the inverse relation between the divisor and the quotient when they are focusing on quantities rather than symbols: a large proportion of 6- and 7- year olds understands, for example, that the more people sharing a cake (or a certain number of sweets), the less each one receives. However, this understanding does not translate immediately into knowledge of how fractions can be put in order of magnitude. Hart et al. (1985) asked students to place in order of magnitude the fractions ¼, ½, 1/100 and 1/3. This could be an easy item because the numerator is constant across fractions, but only about 2/3 of the students in the age range 11-13 ordered these fractions correctly.

In conclusion, rational numbers are required for representing quantities that arise in division situations, rather than as the result of counting. So, in order to understand the connection between the quantities represented by rational numbers and fraction symbols, students must understand the relations between the three quantities in a division situation. The same fraction may represent different quantities when they are fractions of different wholes. Two different fractions represent the same quantity when the relationship between the numerator and the denominator is the same, although the numerator and denominator of the two fractions are different. For fractions of the same numerator, the larger the denominator, the smaller the quantity represented. Finally, the generalization of the use of fraction symbols between part-whole and quotient situations is not obvious to students, and insights developed in quotient situations may not be transferred to part-whole situations, and vice-versa.

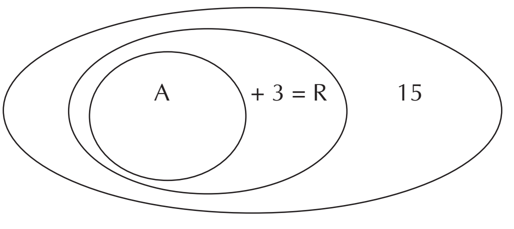
Solving Arithmetic Problems

Much attention in research about solving arithmetic problems has focused on learning to calculate with multi-digit numbers. This valuable research (e.g. Brown & VanLehn, 1982; Resnick, 1982) taught us much about the principled way in which children approach computations, even when they make errors. This research will not be discussed here because the levels of difficulty of calculation with the different types of multi-digit numbers is well documented: for example, it is known that calculation with regrouping (i.e. carrying or borrowing) is difficult; it is also known that subtracting, multiplying and dividing when there is a zero in the numbers is problematic, but zeros cause fewer problems in addition. So it is not difficult to choose a few computation problems that can offer a good assessment of computation skills. Unfortunately, the best way to teach students how to calculate remains controversial, as well as the very need to teach students the traditional written computation algorithms in the context of modern technological societies (see Nunes, 2008). In spite of this latter problem, this section focuses not on how to do sums but knowing when to do which sums.

In the first 6 to 8 years of primary school, students are taught mathematics that draws on two different types of relations between quantities: additive relations, based on part-whole relations between quantities, and multiplicative relations, based on correspondences (of different types) between quantities. The differences between these two types of relations are best understood if we consider an example. Figure 1.1 presents two problems and identifies the quantities and relations in each one.

Both problems describe a quantity, the total number of books that Rob and Anne have, and the relation between two quantities, Rob's and Anne's books. The relation between the quantities in Problem 1 is described in terms of a part-whole structure, as illustrated in the diagram. Part-whole relations are additive. The relation between the quantities in Problem 2 is described in terms of one-to-many correspondence, as illustrated in the diagram; these are multiplicative relations.

(1) Together Rob and Anne have 15 books (quantity). Rob has 3 more books than Anne (or Anne has 3 books fewer than Rob) (relation). How many books does each one have? (quantity)



(2) Together Rob and Anne have 15 books (quantity). Rob has twice the number of books that Anne has (or Anne has half the number of books that Rob has) (relation). How many books does each one have? (quantity)

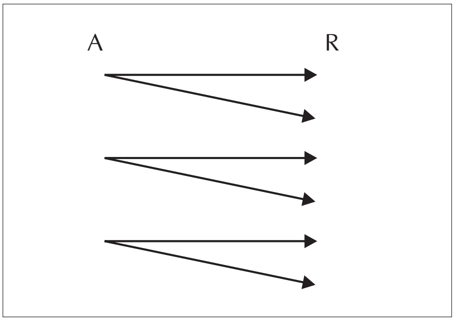


Figure 1.1 A schematic representation of relationships between quantities in additive and multiplicative situations

A major use of mathematics in problem solving involves the manipulation of numbers in order to arrive at conclusions about the problems without having to operate directly on the quantities: in other words, to model the world. To quote Thompson (1993): "Quantitative reasoning is the analysis of a situation into a quantitative structure – a network of quantities and quantitative relationships … . A prominent characteristic of reasoning quantitatively is that numbers and numeric relationships are of secondary importance, and do not enter into the primary analysis of a situation. What is important is relationships among quantities" (p. 165). If students analyze the relationships between quantities in a way that represents the situation well, the mathematical model they build of the situation will be adequate, and the calculations that they implement will lead to correct predictions. If they analyze the relationships between quantities in a way that distorts the situation, the model they build of the situation will be inadequate, and the calculations that they implement will lead to incorrect predictions.

Some situations are immediately understood as additive or multiplicative, and young children, aged 5 and 6, can solve problems about these situations even before they know how to calculate. They use different actions in association with counting to solve these problems. Their actions reveal the way in which they establish relations between the quantities.

A great deal of research (e.g. Brown, 1981; Carpenter, Hiebert, & Moser, 1981; Carpenter & Moser, 1982; De Corte & Verschaffel, 1987; Kintsch & Greeno, 1985; Fayol, 1992; Ginsburg, 1977; Riley, Greeno, & Heller, 1983; Vergnaud, 1982) shows that pre-school children use the appropriate actions when solving problems that involve changes in quantities by addition or subtraction: to find the answers to these problems, they put together and count the items, or separate and count the relevant set. Very few pre-school children seem to know addition and subtraction facts; yet, when they are given the size of two parts, and asked to tell the size of the whole, their rate of correct responses is above 70%, if the numbers are small and they have no difficulty with counting. This is probably not surprising to most people.

However, most people seem surprised when they find out that such young children also show rather high rates of success in multiplication and division problems when they can use objects to help them answer the questions. Carpenter, Ansell, Franke, Fennema, and Weisbeck (1993) gave multiplicative reasoning problems to U.S. kindergarten children involving correspondences of 2 : 1, 3 : 1 and 4 : 1 between the sets (e.g. 2 sweets inside each cup;

how many sweets in 3 cups?). They observed 71% correct responses to these problems. Becker (1993) observed 81% correct responses to multiplicative reasoning problems among 5-year-olds in U.S. kindergartens, when the ratios between quantities were 2 : 1 and 3 : 1.

So, when objects are available for manipulation, young children distinguish easily between the actions they need to carry out to solve simple additive and multiplicative problems. However, the level of difficulty of different types of problems varies within both additive and multiplicative reasoning problems. Vergnaud (1982) argued that what makes many arithmetic problems difficult is not the numerical calculation that students need to carry out but the difficulty of understanding the relations involved in the problem situations. Vergnaud refers to this aspect of problem solving as the relational calculus, which he distinguishes from the numerical calculus – i.e. from the computation itself. In the subsequent sections, we discuss first the difficulties of relational thinking in the domain of additive reasoning and then in the domain of multiplicative reasoning.

Additive Reasoning Problems

Different researchers (e.g. Carpenter, Hiebert, & Moser, 1981; Riley, Greeno, & Heller, 1983; Vergnaud, 1982) proposed very similar classifications for the simplest forms of problems involving addition and subtraction. The basis of these classifications is the type of relational calculation involved. Three groups of problems are identified using this approach. In the first group problems, known as combine problems, were included problems about quantities which were combined (or separated) but not changed (e.g. Paul has 3 blue marbles and 6 purple marbles; how many marbles does he have altogether?). The second group, known as change problems, included problems that involved transformations from initial states resulting in final states (e.g. Paul had 6 marbles; he lost 4 in a game; how many does he have now?). The third group, known as comparison problems, included problems in which relational statements are involved (e.g. Mary has 6 marbles; Paul has 9 marbles; how many more marbles does Paul have than Mary?). The question “how many more marbles does Paul have than Mary” is a question about a relation rather than a quantity. It can be reformulated as “how many fewer marbles does Mary have than

Paul?" Relations have a converse ("how many more" has the converse "how many fewer?"); quantities do not.

The research carried out about these different types of problems showed that combine problems and change problems in which the initial state was known were the easiest problems. Children aged about 6 perform at ceiling level in these types of problems. However, the simplest comparison problems are still difficult for many 8 year olds whereas the most difficult ones, which involve thinking of the converse statement about the comparison, are still challenging for many students in the age 10–11 years. For example, Verschaffel (1994), working with a small sample of students in Belgium reported that if students were given the problem "Charles has 34 nuts. Charles has 15 nuts less than Anthony. How many nuts does Anthony have?", about 30% subtracted 15 from 34 and answered incorrectly. Lewis and Mayer (1987) reported that this error was still presented among U.S. college students, aged 18 years or older, but to a lesser degree (about 16%).

Combine problems always involve quantities and are relatively simple even when the number representing the quantities in the problem is increased. However, change problems involve transformations; combining transformations is more difficult than combining quantities and analyzing transformations is more difficult than separating quantities. For example, consider the two problems below, the first about combining a quantity and a transformation and the second about combining two transformations.

(1) Pierre had 6 marbles. He played one game and lost 4 marbles. How many marbles did he have after the game?

(2) Paul played two games of marbles. He won 6 in the first game and lost 4 in the second game. What happened, counting the two games together?

French children, who were between pre-school and their fourth year in school, consistently performed better on the first than on the second type of problem, even though the same arithmetic calculation (6 – 4) is required in both problems. By the second year in school, when the children are about 7-years-old, they achieved 80% correct responses in the first problem, and they only achieve a comparable level of success in the second problem two years later, when they were about 9 years. So, combining transformations is more difficult than combining a quantity and a transformation.

Three studies can be used to illustrate the difficulty of thinking about relations between quantities, two coming from a quantitative and one from a qualitative tradition.

This first example comes from the Chelsea Diagnostics Mathematics Tests (Hart, Brown, Kerslake, Kuchermann, & Ruddock, 1985), which includes three problems about relations. All three problems have distances as the problem content: distance is not a measure but a relation between two points. The simplest problem is "John is cycling 8 miles home from school. He stops at a sweet shop after 2 miles. How do you work out how much further John has to go?" The question was a multiple choice one, and included three possible answers involving addition and subtraction: 8 – 2, 2 – 8, and 2 + 6. The other four choices involved operations with either the multiplication or division signs. A total of 874 students participated in this study, whose ages were in the ranges 10–11, 11–12 and 12–13 years. The rate of correct responses did not show any increase between 10–11 and 12–13 years, and varied around 68% correct. The other two problems that were of a similar type showed a similar leveling of performance at about 70%. (One problem which had two correct answers showed a slightly higher percentage of correct responses, reaching 78% for the 11–12 year olds.)

The second example involves the use of positive and negative numbers and relations to solve a problem. Our own work (not published in this level of detail yet) illustrates this. The data came from a longitudinal study with two cohorts; both cohorts were tested when they were on average about 10 years 7 months (*N* = 7,981) and the first cohort was tested again in the same items when they were on average 12 years 8 months (*N* = 2,755).

The problem was about pinball games, in which the player's score depends on the number of balls placed in different parts of the board (see Figure 1.2). For each ball in the treasure zone, the player wins one point; for each ball in the skull zone, the player loses one point; no points are awarded for balls lost in the bottom. Obtaining the score for each game is a relatively simple question when all the points are positive: about 90% of the students correctly give the score for Game 3. The rate of correct responses goes down to 48% and 66%, respectively for the 10–11 and 12–13 age groups, when the player lost points. However, combining information about the end result with the information about these two games in order to indicate the player's score in the first game is a much more difficult task: only 29% of the students in the 10–11 year old group and 46% of the students in the 12–13 year old group were successful here. Because the numbers in the problems are small, it is not possible to explain the problem difficulty by the difficulty of the numerical calculus: the difficulty must be connected to the relational calculus. In the pinball game, positive and negative numbers have to be combined, and a relation between the score in two games and the final score must be used in order to infer what the score in the first game must have been.

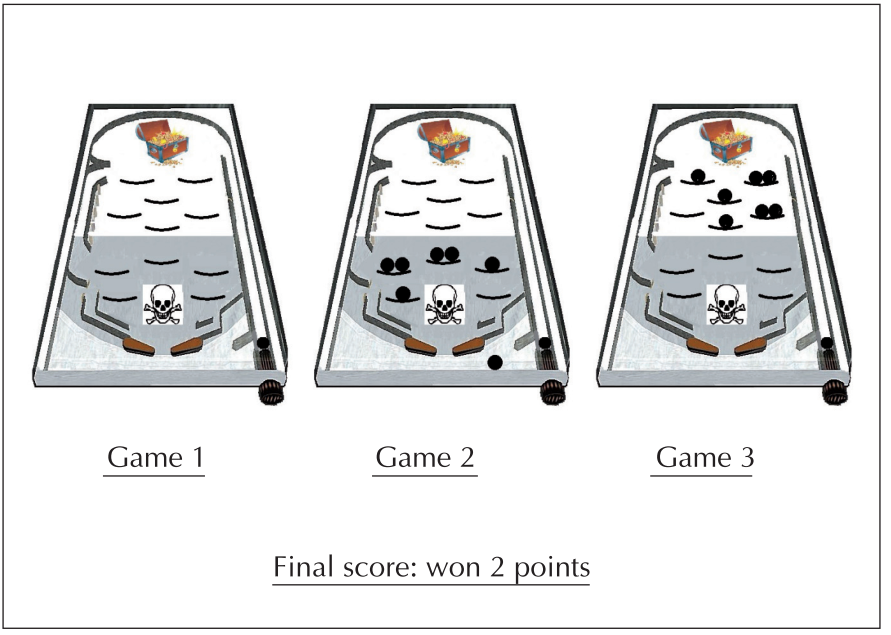


Figure 1.2 An example of a problem based on the pinball game

A few studies about directed numbers (positive and negative numbers) have been carried out in the past, which show that, when all the numbers have the same sign (i.e. are all positive or negative), students treat them as natural numbers, and then assign to them the sign that they had. But combining information from negative and positive numbers requires much more relational reasoning. Marthe (1979), for example, found that only 67% of the students in the age group 14–15 years were able to solve the problem "Mr. Dupont owes 684 francs to Mr. Henry. But Mr Henry also owes money to Mr. Dupont. Taking everything into account, it is Mr. Dupont who must give back 327 francs to Mr. Henry. What amount did Mr. Henry owe to Mr. Dupont?"

Finally, the third example is provided by Thompson's (1993) qualitative analysis of the difficulties that students encounter in distinguishing between relations and quantities in a study with 7- and 9-year olds. He analyzed students' reasoning in complex comparison problems which involved at least three quantities and three relations. His aim was to see how children interpreted complex relational problems and how their reasoning changed as they tackled more problems of the same type. To exemplify his problems, the first problem is presented here: „Tom, Fred, and Rhoda combined their apples for a fruit stand. Fred and Rhoda together had 97 more apples than Tom. Rhoda had 17 apples. Tom had 25 apples. How many apples did Fred have?" (p. 167). This problem includes three quantities (Tom's, Fred's and Rhoda's apples) and three relations (how many more Fred and Rhoda have than Tom; how many fewer Rhoda has than Tom; a combination of these two relations). He asked six children who had achieved different scores in a pre-test (three with higher and three with middle level scores) sampled from two grade levels, second (aged about 7) and fifth (aged about 10) to discuss six problems presented over four different days. The children were asked to think about the problems, represent them and discuss them.

On the first day the children went directly to trying out calculations and treated the relations as quantities: the statement "97 more apples than Tom" was interpreted as "97 apples". This led to the conclusion that Fred has 80 apples because Rhoda has 17. On the second day, working with problems about marbles won or lost during the games, the researcher taught the children to use representations for relations by writing, for example, "plus 12" to indicate that someone had won 12 marbles and "minus 1" to indicate that someone had lost 1 marble. The children were able to work with these representations with the researcher's support, but when they combined two statements, for example minus 8 and plus 14, they thought that the answer was 6 marbles (a quantity), instead of plus six (a relation). So at first they represented relational statements as statements about quantities, apparently because they did not know how to represent relations. However, after having learned how to represent relational statements, they continued to have difficulties in thinking only relationally, and unwittingly converted the result of operations on relations into statements about quantities. Yet, when asked whether it would always be true that someone who had won 2 marbles in a game would have 2 marbles, the children recognized that this would not necessarily be true. They did understand that relations and quantities are different but they interpreted the result of combining two relations as a quantity.

Unfortunately, Thompson's study does not include quantitative results from which we could estimate the level of difficulty of this type of problem at different age levels but it can be reasonably hypothesized that students at age 13–15 have not yet mastered problems where many relations and quantities must be combined in order to solve the problem.

A brief summary of how students progress in additive reasoning can be gleamed from the literature.

(1) From a very early age, about 5 or 6 years, children can start to use counting to solve additive reasoning problems. They can use the schemas of joining and separating to solve problems that involve combining quantities, separating quantities, or transforming quantities by addition and subtraction.

(2) It takes about two to three years for them to start using these actions in a coordinated fashion, forming a more general part-whole schema, which allows them to solve simple comparison problems.

(3) Combining transformations and relations to solve problems (such as combining two distances to find the distance between two points) continues to be difficult for many students. The CSMS study shows a leveling off of rates of correct responses about age 13; older age groups were not tested in these problems.

(4) The same additive relation can be expressed in different ways, such as "more than" or "less than". When students need to change the relational statement into its converse in order to implement a calculation, they may fail to do so.

(5) Combining positive and negative numbers seems to remain difficult until the age of 14 (no results with 15 year olds were reviewed). The rate of correct responses in some of the problems does not surpass 50%.

Multiplicative Reasoning Problems

Research on multiplicative reasoning problems has produced less agreement in the classification of problem types. The different classifications seem to be based on different criteria rather than on conceptual divergences about the nature of multiplicative problems. We do not attempt to reconcile these differences here but refer to them in footnotes as we describe the development of multiplicative reasoning. We will adopt here Vergnaud's terminology and refer to others as required.

Vergnaud (1983) distinguished between three types of multiplicative reasoning problems:

(1) isomorphism of measures problems, which involve two measures connected by a fixed ratio (Brown, 1981, refers to these problems as ratio or rate);
(2) multiple proportions, in which more than two measures are proportionally related to each other;
(3) product of measures, in which two measures give rise to a third one, the product of the two (Brown, 1981, refers to these as Cartesian problems).[1]

Isomorphism of measures problems include the simple problems described earlier on, which young children can solve by setting items in correspondence. These are the most commonly used type of proportions problems in school; they involve a fixed ratio between two measures. Common examples of such problems are number of people for whom a recipe is prepared and amount of ingredients; number of muffins one makes and amount of flower; quantity purchased and price paid. The level of difficulty of these problems is influenced by the availability of materials that can be used to represent the correspondences between the measures, the ratio between the measures (2:1 and 3:1 are much easier than other ratios), the presence of the unit value in the problem (3:1, for example, is easier than 3:2), and the values used in the problem (if the unknown is either a multiple or a divisor of the known value in the same measure, it is possible to solve the problem using scalar reasoning or within-quantity calculations, the most commonly used by students). In some countries (e.g. France; see Ricco, 1982; Vergnaud, 1983), students are taught a general algorithm (e.g. finding the unit value; the Rule of Three) that can be used to solve all proportions problems, but students often use other methods when proportions problems are presented amongst other problems with different structures (Hart, 1981; Ricco, 1982; Vergnaud, 1983). These student-designed methods have been identified under different terminologies but are remarkably similar across

1 The term *measure* is used here rather than *quantity* because some quantities may be measured differently and problems about these quantities would thus end up in different categories. For example, if the area of a parallelogram is measured with square units, the calculation of its area will be an example of isomorphism of measures problems: number of units in a row times number of rows. If the area is measured using linear units, the calculation is a product of measures, as a square unit such a $1cm^2$ will be the product of the two linear units, 1cm x 1cm.

countries. They involve parallel transformations within each measure (e.g. dividing each measure by two), a move which preserves the ratio between the measures, and a progressive approximation to the answer, without losing sight of the correspondences between measures. Hart's (1981) well known example of the onion soup recipe for 8 people, which has to be converted into a recipe for 6 people, illustrates this approach to solution well. Students tend to calculate what ingredients would be required for 4 people (i.e. half of the ingredients for 8 people), then what would be required for 2 people, and then add the ingredients for 4 with those for 2 people – thus finding the solution for 6 people.

Systematic comparisons using carefully matched values across problems (see, for example, Nunes, Schliemann, & Carraher, 1993) show that the students approach proportions problems more often by thinking of the relations within the same measure than of the relations between the two different measures in the problem. For example, in the same onion soup recipe, the ratio pints of water per person was 1:4. Students could calculate this ratio from the recipe for 8 people and find how much water for 6 people, but this solution was not reported by Hart.[2]

In summary, in the assessment of younger children's competence in isomorphism of measures problems one can vary the level of problem difficulty by varying the materials available for representing the problem; in the assessment of older children, one can vary the level of problem difficulty by using numbers that make either within-quantity or between-quantities calculation easier.

Multiple proportions problems involve a situation in which more than two measures have a fixed ratio. Vergnaud proposed as an example problem the question of finding the amount of milk produced in a farm, which is related to the number of cows in the farm and the number of days. Multiple proportion problems are more difficult than simple isomorphism of measures problems, as they involve handling more information and carrying out more calculations. It is, however, not clear whether they pose new conceptual challenges for students.

2 Nesher (1988) and Schwartz (1988) suggest that dividing one quantity by another, the move required to calculate the ratio of water to people, changes the referent of the number: instead of thinking of 2 pints of water, one must think of 1 pint per 4 people. They attribute to this transformation of the referent the higher level of difficulty of some problems. This leads them to classify multiplication problems using a different schema, which is not discussed here.

Proportional reasoning is one of the most crucial areas of mathematics education, as it is the basis of understanding in several further domains of mathematics. It is applied in other school subjects, first of all, in science. Several everyday life contexts require handling of proportions as well. Therefore, a number of large-scale assessment projects explored its development (Kishta, 1979; Schröder at al., 2000; Misailidou, & Williams, 2003; Boyera, Levinea, & Huttenlochera, 2008; Jitendra at al., 2009). In a cross-sectional assessment where its development was examined in relation to analogical and inductive reasoning, it was found that at grade five only 7% and at grade seven 20% of students was able to solve a simple proportional task (Csapó, 1997).

Product of measures problems involve more a composition of two measures to form a third measure: for example, the number of T-shirts and number of skirts a girl has can be composed to give the number of different outfits that she can wear; the number of different colored cloths and the number of emblems determines the number of different flags that you can produce; the different types of bread and the number of different fillings in a delicatessen can be combined to form different types of sandwiches. Thus product of measures problems involve a qualitative multiplication – i.e. the combination of different qualities in a multiple classification – as well as a quantitative multiplication. Product of measures problems are significantly more difficult than isomorphism of measures problems (Brown, 1981; Vergnaud, 1983). They are a significant part of multiplicative reasoning and thus should be evaluated in the assessment of students' multiplicative reasoning.

The development of students' understanding of product of measures problems is not an all-or-nothing matter. These problems can be simplified in presentation, by using suggestions of how the combinations work in a step-wise presentation: with one skirt and 3 blouses, how many different outfits can you make; if you buy a new skirt, how many new combinations can you make? When product of measures problems are presented in a step-wise manner, the rate of correct responses increases significantly (Nunes & Bryant, 1996). Figure 1.3 shows an example of a product of measures problem in which the first two combinations are presented; students might easily find the remaining combinations, and count the total possible number. Students aged 11–12 years in the U.K. showed an average of 81% correct responses to this problem; this was significantly better than the rate of 51% correct observed in problems in which only one item was used to suggest how the combinations might work.

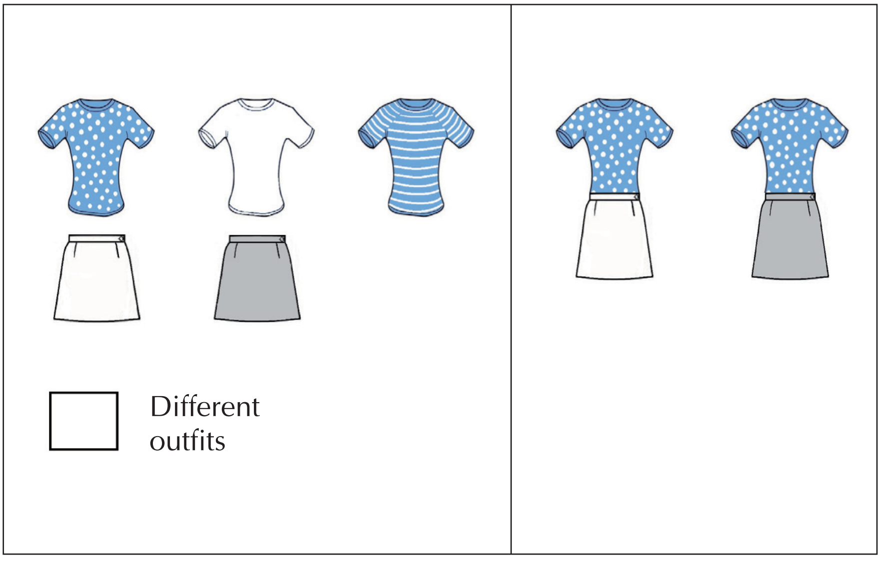


Figure 1.3 An example of product of measures problem in which the first combinations are presented visually

However, students may not necessarily be able to formulate a general rule for finding out the number of possible combinations after this step-wise introduction. The step from thinking that for each new skirt, *x* new outfits to the general rule that, therefore, the number of outfits is the number of skirts times number of outfits, demands considerable effort.

In conclusion, the development of multiplicative reasoning involves two types of correspondences: those exemplified in isomorphism of measures problems, which are quantitative, and those exemplified in product of measures problems, in which an initial, qualitative step based on a multiple classification schema is required. Young children can succeed in isomorphism of measures problems if they can use manipulatives to represent the measures; they solve the problems by counting (i.e. they do not calculate a multiplication) but their reasoning is clearly multiplicative. Product of measures problems are more difficult.

As children progressively master the relational calculations in multiplicative reasoning, they solve a greater variety of problems. However, one challenge remains, even at the end of primary school. Students seem to have greater difficulty in thinking about the between-quantities, functional relation (measured by the rate between variables) than about the within-quantities, scalar relation. Always solving problems using scalar solutions means that students focus only on half of the relationships that are significant in the situation. Teaching that helps students focus on functional relations also helps students make conscious choices of models for problem solving. Students are known to over-use as well as under-use proportional reasoning (e.g. De Bock, Van Dooren, Janssens, & Verschaffel, 2002; De Bock, Verschaffel, & Janssens, 1998; Dooren, Bock, Hessels, Janssens, & Verschaffel, 2004) and also over-use linear relations when asked to represent graphically the relationship between two variables. It is possible that if students became more aware of the functional relations, i.e. the relations between variables, they would be less prone to such errors.

Advancing Cognitive Development Through Mathematics Education

In the preceding sections, we discussed how reasoning and knowledge of numerical systems are inter-related and support each other in mathematics education. Reasoning about quantities is always necessary for understanding how numerical representations work. In this section, the focus is on how good mathematics education can promote a better understanding of relations between quantities and a greater ability to use numbers and other mathematical tools to solve problems.

Research on Advancing Cognitive Development in Mathematics Education

The examples of forms of reasoning about quantities described in the earlier sections are not innate: they develop over time, and this development can be promoted in the context of mathematics education. The influences between mathematical reasoning and learning mathematics in the classroom are reciprocal, in so far as promoting one leads to improvement in the other.

Research by different teams of investigators (Nunes et al., 2007; Shayer & Adhami, 2007) has shown that improving students' thinking about mathematics in the classroom has a beneficial effect on their later mathematics learning. We present briefly here some results from the project by Shayer and Adhami, which included a large number of classrooms and of pupils and extensive professional development for the teachers. Shayer and Adhami's (2007) study included approximately 700 students and their teachers, approximately half of whom were in the control and the other half in experimental classes. The researchers designed a program known as CAME (cognitive acceleration through mathematics education) to be used by grade 5 and 6 teachers and their children (9 to 11 years old), which emphasized reasoning about numerical problems. The teachers participated in two full-day professional development workshops, in which they discussed and re-designed the tasks for their own use. The pupils were assessed in a Piagetian task of spatial reasoning before and after their participation in the program. For the control group, mathematics teaching went on in the business-as-usual format during this period.

In the Piagetian Spatial Relations test (NFER, 1979), children are asked to draw objects in situations chosen to test their notions of horizontality, verticality and perspective. For example, they are asked to draw the water level in jam-jars half-full of water, presented at the various orientations: upright, tilted at 30 degrees off vertical, and on its side. They are also asked to produce a drawing of what they would see if they were standing in the middle of a road consisting of an avenue of trees; the drawing should cover the near distance as well as afar. Assessment of the children's responses consists of seeing how many aspects of the situation, how many relations, they can consider in their drawings. The tasks allow for a classification of the productions as Piaget's early concrete operations (level 2A), if the drawings represent only one relation, or as mature concrete (2B), if they handle two relations.

The tasks included in the program considered many of the issues raised here: for example, with respect to rational numbers, the students were asked to compare the amount of chocolate that recipients in two groups would have; in one group, one chocolate was shared between 3 children and in the other two chocolates were shared between 6 children. The equivalence of fractions could be discussed in this context, which helped the students understand the equivalence in quantities in spite of the use of different fractions to represent the quantities.

Shayer and Adhami (2007) observed a significant difference between the students in the control and the experimental classes, with the experimental classes out-performing the control classes in the Piagetian task as well as in the standardized mathematics assessments designed by the government, and thus completely independent of the researchers.

In summary, mathematical tasks that are well chosen in terms of the demands they place on students' reasoning, and are presented to students in ways that allow them to discuss the mathematical relations as well as the connections between quantities and symbols, contribute to mathematical learning and cognitive development.

Numeric Skills, Additive and Multiplicative Reasoning

The previous sections in this chapter identified different reasoning skills to be developed, as playing a central role in early mathematics education and determining later achievements. This section summarizes the different skills related to this area and outlines their development.

Whole Numbers

In pre-school, children should have the opportunity to learn about the relations between quantities and numbers. The indicators presented here are not exhaustive, but all children must be able to understand that:

(1) if a quantity increases or decreases, the number that represents it changes
(2) if two quantities are equivalent, they are represented by the same number;
(3) if the same number of items is added to and taken away from a set, the number in the set doesn't change;
(4) any number can be composed by the sum of two other numbers;
(5) when counting tokens with different values (money, for example), some tokens count as more than one because their value has to be taken into account.

Children who understand these principles make more progress in learning mathematics throughout the first two years of school than those who do not.

Rational Numbers

Fractions are symbols that represent quantities resulting from division, not from counting. They represent the relation between the terms in a division. Children can start to explore these insights in kindergarten and in the first years in primary school by thinking about division situations. They should be able to understand that:

(1) if two dividends are the same and two divisors are the same, the quotient is the same (e.g. if there are two groups of children with the same number of children sharing the same number of sweets (or sharing cakes of the same size), the children in one group will receive as much as the children in the other group;
(2) if the dividend increases, the shares increase;
(3) if the divisor increases, the shares decrease;

Further insights into the nature of division and fractions can be achieved from about age 8 or 9:

(4) it is possible to share the same dividend in different ways and still have equivalent amounts; the way in which the shares are cut does not matter if the dividend is the same and the divisor is the same;
(5) if the dividends and the divisors are different, the relation between them may still be the same (e.g. 1 chocolate shared by 2 children and 2 chocolates shared by 4 children result in equivalent shares);
(6) these ideas about quantities should be coordinated with the writing of fraction symbols.

These insights about rational numbers enable students to use rational number to represent quantities and can be used to help them learn how to operate with numbers. However, in order to solve problems, students need to learn in primary school to reason about two types of relations between quantities, which lead to mathematizing situations differently: part-whole, which define additive reasoning, and correspondence relations, which define multiplicative reasoning.

Additive Reasoning

The development of additive reasoning involves a growing ability to distinguish quantities from relations and to combine positive and negative relations even without knowledge of quantities. Although there is no single investigation that covers the development of additive relational reasoning, a summary of how students progress in additive reasoning can be abstracted from the literature.

Level 1 Students can solve problems about quantities when these increase or decrease by counting, adding and subtracting. They do not succeed in comparison problems.

Level 2 Students succeed with comparison problems and also in using the converse operation to solve problems. The same additive relation can be expressed in different ways, such as "more than" or "less than". When students need to change the relational statement into its converse in order to implement a calculation, they may fail to do so. At level 2, they are able to convert one relation into its converse in order to solve problems.

Level 3 Students become able to compare positive and negative numbers and to combine two relations to solve problems, but they often do so by hypothesizing a quantity as the starting point for solution. Combining more than two positive and negative relations in the absence of information about quantities remains difficult until the age of 14 (no results with 15 year olds were reviewed). The rate of correct responses in some of the problems does not surpass 50%.

Level 4 Perfect performance in combining additive relations and distinguishing these from multiplicative relations.

Multiplicative Reasoning

Multiplicative reasoning starts with young children's ability to place quantities in one-to-many correspondences to solve diverse problems, including those in which two variables are connected proportionally and sharing situations. It involves the understanding of the notion of proportionality, which includes situations in which there is a fixed ratio between two variables in isomorphism of measures problems, and understanding the multiplicative relation between two measures, which can be combined to form a third one, in product of measures problems.

Level 1 Students can solve simple problems when two measures are explicitly described as being in correspondence and they can use materials to set the variables in correspondence. However, in more complex situations, in which they need to think of this correspondence themselves, they realize that there is a relation between the two variables, so that a change in one variable results in a change in the other one, but may not be able to think of how to systematically establish correspondences between the variables.

Level 2 Students at this level recognize that the two values of the two variables vary together and in the same direction and there is a definite rule behind co-varying. In simple cases and familiar contexts, they recognize the quantitative nature of the relationship but are unable to generalize a rule.

Level 3 At this level students recognize the linear nature of the relationship, and they are able to deal with proportionality in familiar contexts.

Level 4 At this level, students are able to deal with the linear relationship of the two variables in any content and context. They are also able to distinguish linear from non-linear relationships, although they may need to make step by step comparisons when asked to think about novel problems.

The hierarchy outlined here corresponds to the natural order of cognitive development. If teaching always focuses on the level next to the already reached level – individually in cases of every student –, then they possess the mental tools needed for comprehending it. In this way teaching may have optimal developing effect.

Further Areas for Advancing Mathematical Thinking

Beyond the areas of mathematical reasoning discussed in the previous parts of this chapter, there are several further ones to be developed in the early mathematics education. We review some of them in this section, but we do not deal with them in detail. Although the areas of mathematical reasoning are related to each other, the areas of reasoning reviewed in this section are not directly related to numerical reasoning or they are generalized beyond the issues of numbers. Furthermore, fostering their development may also be possible by exercises embedded in other school subjects; therefore, the advancement of reasoning abilities reviewed here may not be narrowed to mathematics education. For example, text comprehension assumes understanding and interpreting operations of propositional logic. Processing complex scientific texts, especially comprehending sophisticated definitions requires handling logical operations. Learning science activates a number of cognitive skills which are developed in mathematics. In this way science education enriches the experiential basis of mathematical reasoning in several aspects, such as seriations, classifications, relations, functions, combinatorial operations, probability and statistics.

Most reasoning abilities listed here were extensively studied by Piaget and his followers. According to their findings, the development of these schemes begins early, well before schooling starts. In the first six years of schooling, in which we are interested, their development is mostly in pre-operational and concrete operational phase, and the formal level can be reached only in the later school years. Therefore, the main task of early mathematics education is to provide students with a stimulating environment to gain experiences for inventing similarities and rules to create their own operational schemes. These systematic experiences should be followed by mastering the mathematical formalisms later. Science education, especially hands-on-science, may contribute to the development by enriching the experiential bases in the early phase, and later at the higher level of abstraction by the application of mathematical tools.

Logical operations and the operations with sets are isomorphic from the mathematical point of view, but the corresponding thinking skills are rooted in different psychological developmental processes. However, their similarities may be utilized in mathematics education. The development of logical operations was examined in detail by Piaget and his co-workers in their classical experiments (Inhelder & Piaget, 1958). Later research has indicated that not only the structure of logical operations determines how people judge the truth of propositions connected by logical operations but the familiarity of context and the actual content of propositions as well (see Wason, 1968, and further research on the Wason task). However, the aim of mathematical education is to help students to comprehend propositions and interpret their meaning determined by the structure of the operations, therefore the conclusions of Piaget's research remain relevant for mathematics education. Furthermore, Piaget's notion that development takes place through several phases and takes time should also be taken into account. As for the operations with sets, for which several tools are available for manipulation, may serve as founding experiences for logical operations. The schemes of concrete, manipulative operations carried out by objects may be interiorized and promote the development of operations of propositional logic. On the other hand, developing propositional logic is a broader educational task, in which pre-primary education should play an essential role, as well as several further school subjects. In the later phases teaching of other school subjects may contribute to fostering the development by analyzing the structure of logical operations and by highlighting the relationships between structure and meaning. There are several broadly used instruments for assessing the development of logical operation (see e.g. Vidákovich, 1998).

Relations appear in several areas of mathematics education. Reasoning with some relations has been discussed in the previous sections, and several further operations involving relations were examined by Piaget, too (Piaget & Inhelder, 1958). Among others these are seriations and class inclusions. The construction of series plays a role in the development of proportional reasoning discussed earlier and may contribute to several broader reasoning abilities, such as analogical and inductive reasoning (see Csapó, 1997, 2003). Recognizing rules in series and correspondences in classifications develop skills of rule induction and contribute to the concept of mathematical function. As the mathematical conception of function is a result of multiple abstractions, a solid experiential base in essential for further learning. Relations may be represented and visualized in several ways. Understanding the correspondence between different representations and carrying out transformations between representations may foster analogical reasoning. Developing the skills related to multiple representations is also a task of mathematics education.

From a mathematical point of view, combinatorics, probability and statistics are closely related, but the corresponding psychological developmental processes originate from different points. Spontaneous stimuli coming from an average environment cannot connect these different ideas; only systematic mathematics teaching may lead to establishing connections among them at a more mature level.

Combinatorial problems may be classified into two main groups. In enumeration tasks students are expected to create all possible constructs out of given elements, permitted by conditions or situations. Some problems of this type may be solved by combinatorial reasoning. In the other group are the computation problems, when the number of possible constructs should be calculated, which, in general case, can be solved only after systematic mathematics education. We have already discussed some aspects of combinatorial reasoning concerning the multiplicative reasoning. Combinatorial structures play a central role in Piaget's theory of development of propositional logic, and he also examined the development of some combinatorial operation (Piaget, és Inhelder, 1975). Several further research projects explored the structure and development of combinatorial thinking and the possibilities of fostering it both in mathematics and in other school subjects. (Kishta, 1979; Csapó, 1988, 2001, 2003; Schröder, Bödeker, Edelstein, & Teo, 2000; Nagy, 2004). An analysis identified 37 combinatorial structures, according to the number of variables, the number of values of variables, and the number of constructs to be created that may be handled. On the basis of these operational structures, test tasks can be devised. The empirical investigation based on these tasks revealed that some children were able to solve every task by age 14, but most of them were able to deal only with the most basic operations (Csapó, 1988). The charge of early mathematical education is the stimulation of the development of combinatorial reasoning by well structured exercises, while enumeration tasks may be embedded in other school subjects as well, which can also foster combinatorial reasoning (Csapó, 1992). Nevertheless, preparing the formalization of reasoning processes and teaching computational problems can be done only in mathematics.

The development of the idea of chance and probability begins early (Piaget & Inhelder, 1975), but without systematic stimulation most children reach only a basic level. Understanding nondeterministic relationships and correlations is especially difficult (Kuhn, Phelps, & Walters, 1985; Bán, 1998; Schröder, Bödeker, Edelstein, & Teo, 2000). Development of probabilistic reasoning may be promoted in early mathematics by illustrating chance, while other school subjects (e.g. biology) may present probabilistic phenomena to enrich the experiential basis of development. Later, systematic exercises may prepare the introduction of formal interpretation of probability as ratio of the number of occurrences of different events.

A further area, spatial reasoning is rooted in other developmental processes, different from that of numeric reasoning, and is related to measures and numbers in later developmental phases in the framework of systematic geometry education. Piaget explored the development of spatial reasoning mostly trough the representation of space in children's drawings. According to his results, early development may be characterized by a topologic view, when first the connecting points of lines are correct on drawings, but shapes are distorted. The shapes drawn by children get further differentiated during the second stage (age 4–7). In the third stage children draw shapes and forms which are correct in Euclidean terms (Piaget & Inhelder, 1956). Spatial reasoning may be fostered in the early mathematics education by systematic exercises of studying two and three dimensional forms. Then, students may be encouraged to infer that shapes have properties, and similarities and differences between shapes may be characterized by these properties. Later, properties may be precisely defined in the framework of geometry teaching.

Spatial reasoning may be fostered in the framework of teaching drawing and art education as well. A number of different instruments have been devised for the assessment of representation of space in the framework of art education (see e.g. Kárpáti & Gaul, 2011; Kárpáti & Pethő, 2011).

Assessing Cognitive Development in Mathematics

One of the major points in the preceding discussion was the significance of reasoning for understanding number system and for understanding how to use mathematics to model the world. A second thread in the discussion is that most insights that students have to develop in mathematics do not develop in a single step. For this reason, assessments of cognitive development in the context of mathematics should be designed in ways that place little demands on computation in comparison to relational calculations. Computation can, and should, be assessed on its own merits.

Content of Assessment

Reasoning skills that are predictive of mathematics achievement must be at the core of assessments of cognitive development in mathematics. Different predictive studies have shown that, in the early years, children's performance in tasks that assess their knowledge of correspondences, seriation, additive composition, and the inverse relation between addition and subtraction predict their performance later, in standardized tests of mathematics achievement (Nunes et al., 2007; van de Rijt, van Luit, & Pennings, 1999). Early number skills, sometimes referred to as number sense, is also a predictor of mathematics achievement (Fuchs, Compton, Fuchs, Paulsen, Bryant, & Hamlet, 2005).

Measures of early number knowledge include knowing how to write and read numbers, how to compare the magnitude of written and oral numbers, and some computation bonds. As far as we know, only one study (Nunes et al., 2011) has compared the relative importance of number skills and mathematical reasoning in the prediction of mathematical achievement in a large scale longitudinal analysis. This comparison also included the cognitive functions of attention and memory in the analysis, and a measure of IQ. All predictors were assessed when the students were in the 8–9 year age range; the measures of mathematical achievement were independent standardized measures obtained by the school when they were 11–12 and 13–14 years. All the predictors made significant and independent contributions to variation in mathematics achievement at ages 11–12 and 13–14. For both time points, mathematical reasoning made a stronger contribution than attention and memory and also than numerical skills. This analysis of the relative importance of mathematical reasoning and number skills suggests that, if time limits are significant, it is more important to assess mathematical reasoning than numerical skills.

Forms of Assessing Thinking Abilities in Mathematics

The design of mathematical assessments is inevitably related to students' general ability to understand instructions and other verbal skills. However, it is possible to minimize the influence of reading skills by designing assessments that use drawings to help students imagine the problem situations. Drawings also allow students to use different approaches in establishing the numerical value of their answer: they can often analyse the drawings (e.g. divide something in two to help them imagine the value of half the quantity) and even count in order to determine the answer. As long as their analysis of the situation is correct, they have a better chance of quantifying the answer correctly than if the problems were presented only in writing. Researchers in the Freudenthal Institute pioneered this approach to assessment (see, for example, van den Heuvel-Panhuizen, 1990), which gives valuable information about students' relational calculation skills.

As discussed earlier on, mathematical insights develop over time. Ideally, one should include in assessments different levels of demands on the reasoning skills. These can be varied by using different forms of representation, different situations, as well as different values. The preceding review suggests how these variations can be attained within the assessment of the same type of concept.

Summary

In this chapter we have reviewed some major areas of the development of mathematical reasoning. We have focused our attention on those psychological questions which are the most crucial ones from the point of view of early cognitive development. We highlighted those thinking abilities that may be developed almost exclusively in the framework of mathematics education. Among these areas are the reasoning about measures and numbers and the development of the relations among concepts and skills.

We have emphasized that developing mathematical thinking differs from mastering mathematical knowledge. The beginning phase of schooling should focus on fostering the development of mathematical thinking, as without proper reasoning skills mathematics cannot be comprehended.

We have discussed four areas in more detail: reasoning about whole numbers, rational numbers, additive and multiplicative reasoning. These are especially important as they form the foundations for later mathematics learning. Results of several research projects indicated the predictive power of these thinking abilities; the early levels assessed in these areas predict the achievements measured later.

We have also indicated that there are several further important components of mathematical reasoning. They can be developed and assessed in similar ways to those that were described in more detail.

We have emphasized at several points that the development of mathematical thinking is a slow and long process taking several years. However, several research and development projects have shown that systematic stimulating exercises can accelerate development. These exercises can result in improvement of thinking only if they are carefully matched to the actual developmental level of students. Therefore, in mathematics education, the sequence of developmental stimuli is especially important. A complex thinking process can develop successfully only if the preceding phase has already been completed and the component skills developed.

Consequently, for the development of mathematical thinking, it is inevitable that teachers know well the actual developmental level of their students. This allows them to adjust teaching individually to the need of every student. In order to meet this need, the instruments of diagnostics assessment should cover the developmental process of mathematical thinking.

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